

1D QUANTUM HARMONIC OSCILLATOR PERTURBED BY A POTENTIAL WITH LOGARITHMIC DECAY

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ABSTRACT. In this paper we prove an infinite dimensional KAM theorem, in which the assumptions on the derivatives of perturbation in [22] are weakened from *polynomial decay* to *logarithmic decay*. As a consequence, we apply it to 1d quantum harmonic oscillators and prove the reducibility of a linear harmonic oscillator, $T = -\frac{d^2}{dx^2} + x^2$, on $L^2(\mathbb{R})$ perturbed by a quasi-periodic in time potential $V(x, \omega t; \omega)$ with *logarithmic decay*. This entails the pure-point nature of the spectrum of the Floquet operator K , where

$$K := -i \sum_{k=1}^n \omega_k \frac{\partial}{\partial \theta_k} - \frac{d^2}{dx^2} + x^2 + \varepsilon V(x, \theta; \omega),$$

is defined on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^n)$ and the potential $V(x, \theta; \omega)$ has logarithmic decay as well as its gradient in ω .

1. INTRODUCTION AND MAIN RESULTS

1.1. Statement of the Results. In this paper we consider the linear equation

$$i\partial_t u = -\partial_x^2 u + x^2 u + \varepsilon V(x, \omega t; \omega) u, \quad u = u(t, x), \quad x \in \mathbb{R}, \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter and the frequency vector ω of forced oscillations is regarded as a parameter in $\Pi := [0, 2\pi)^n$. We assume that the potential $V : \mathbb{R} \times \mathbb{T}^n \times \Pi \ni (x, \theta; \omega) \mapsto \mathbb{R}$ is C^3 smooth in all its variables and analytic in θ where $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ denotes the n -dimensional torus. For $\rho > 0$, the function $V(x, \theta; \omega)$ analytically in θ extends to the domain

$$\mathbb{T}_\rho^n = \{(a + bi) \in \mathbb{C}^n / 2\pi\mathbb{Z}^n : |b| < \rho\}$$

as well as its gradient in ω and satisfies

$$|V(x, \theta; \omega)|, |\partial_{\omega_j} V(x, \theta; \omega)| \leq C(1 + \ln(1 + x^2))^{-2\beta}, \quad (1.2)$$

$$|\partial_x V(x, \theta; \omega)|, |\partial_x \partial_{\omega_j} V(x, \theta; \omega)| \leq C, \quad (1.3)$$

$$|\partial_x^2 V(x, \theta; \omega)|, |\partial_x^2 \partial_{\omega_j} V(x, \theta; \omega)| \leq C, \quad (1.4)$$

where $(x, \theta; \omega) \in \mathbb{R} \times \mathbb{T}_\rho^n \times \Pi$, $\beta \geq 2(n+2)$, $j = 1, \dots, n$ and $C > 0$.

Theorem 1.1. *Assume that V satisfies (1.2) - (1.4) and $\beta \geq 2(n+2)$. Then there exists ε_0 such that for all $0 \leq \varepsilon < \varepsilon_0$ there exists $\Pi_\varepsilon \subset [0, 2\pi)^n$ of positive measure and asymptotically full measure: $\text{Meas}(\Pi_\varepsilon) \rightarrow (2\pi)^n$ as $\varepsilon \rightarrow 0$, such that for all $\omega \in \Pi_\varepsilon$, the linear Schrödinger equation (1.1) reduces, in $L^2(\mathbb{R})$, to a linear equation with constant coefficients (with respect to the time variable).*

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Similar as [22], the above theorem has two direct corollaries. As a preparation we define the harmonic oscillator $T = -\frac{d^2}{dx^2} + x^2$ and its related Sobolev space. Let $p \geq 2$ and denote $\ell^{2,p}$ be the Hilbert space of all real $w = (w_j)_{j \geq 1}$ with

$$\|w\|_p^2 = \sum_{j \geq 1} j^p |w_j|^2 < \infty.$$

The operator T has eigenfunctions $(h_j)_{j \geq 1}$, so called the Hermite functions, which satisfy

$$Th_j = (2j-1)h_j, \quad \|h_j\|_{L^2(\mathbb{R})} = 1, \quad j \geq 1, \quad (1.5)$$

and form a Hilbertian basis of $L^2(\mathbb{R})$. Let $u = \sum_{j \geq 1} u_j h_j$ be a typical element of $L^2(\mathbb{R})$. Then $(u_j) \in \ell^{2,p}$ if and only if

$$u \in \mathcal{H}^p := D(T^{p/2}) = \{u \in L^2(\mathbb{R}) : x^{\alpha_1} \partial_x^{\alpha_2} u \in L^2(\mathbb{R}) \text{ for } 0 \leq \alpha_1 + \alpha_2 \leq p\}.$$

For a function $f \in \mathcal{H}^p(\mathbb{R})$ we define

$$\|f\|_{\mathcal{H}^p}^2 = \sum_{0 \leq \alpha_1 + \alpha_2 \leq p} \|x^{\alpha_1} \partial_x^{\alpha_2} f\|_{L^2}^2 < \infty.$$

Then we have

Corollary 1.2. *Assume that V and $\partial_{\omega_j} V$, $j = 1, \dots, n$ are C^∞ in x and there exists a constant $C > 0$ such that for all $\nu \geq 1$, $x \in \mathbb{R}$ and $|\Im \theta| < \rho$,*

$$\begin{aligned} |V(x, \theta; \omega)|, |\partial_{\omega_j} V(x, \theta; \omega)| &\leq C(1 + \ln(1 + x^2))^{-2\beta}, \\ |\partial_x^\nu V(x, \theta; \omega)|, |\partial_x^\nu V_{\omega_j}(x, \theta; \omega)| &\leq C. \end{aligned}$$

Let $p \geq 0$ and $u_0 \in \mathcal{H}^p$ and $\beta \geq 2(n+2)$. Then there exists $\varepsilon_0 > 0$ so that for all $0 \leq \varepsilon < \varepsilon_0$ and $\omega \in \Pi_\varepsilon$, there exists a unique solution $u \in \mathcal{C}(\mathbb{R}, \mathcal{H}^p)$ of (1.1) so that $u(0) = u_0$. Moreover, u is almost-periodic in time and we have the bounds

$$(1 - \varepsilon C) \|u_0\|_{\mathcal{H}^p} \leq \|u(t)\|_{\mathcal{H}^p} \leq (1 + \varepsilon C) \|u_0\|_{\mathcal{H}^p}, \text{ with } t \in \mathbb{R},$$

for some $C = C(p, \omega)$.

Consider the Floquet operator on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^n)$

$$K := -i \sum_{k=1}^n \omega_k \frac{\partial}{\partial \theta_k} - \frac{d^2}{dx^2} + x^2 + \varepsilon V(x, \theta; \omega), \quad (1.6)$$

then we have

Corollary 1.3. *Assume that V satisfies the same conditions as in Theorem 1.1 and $\beta \geq 2(n+2)$. There exists $\varepsilon_0 > 0$ so that for all $0 \leq \varepsilon < \varepsilon_0$ and $\omega \in \Pi_\varepsilon$, the spectrum of the Floquet operator K is pure point.*

1.2. Related results. As in [1] the equations (1.1) can be generalized into a time-dependent Schrödinger equation

$$i\partial_t \psi(t) = (A + \varepsilon P(\omega t)) \psi(t), \quad (1.7)$$

where A is a positive self-adjoint operator on a separable Hilbert space \mathcal{H} and the perturbation P is an operator-valued function from \mathbb{T}^n into the space of symmetric operators on \mathcal{H} . The Floquet operator associated with (1.7) is defined by

$$K_F := -i\omega \cdot \partial_\theta + A + \varepsilon P(\theta) \quad \text{on } \mathcal{H} \otimes L^2(\mathbb{T}^n).$$

It is well-known that long-time behavior of the solution $\psi(t)$ of the time-dependent Schrödinger equation (1.7) is closely related to the spectral properties of the Floquet operator K_F (see Wang [42]). It has been proved in [2, 7, 8, 9, 20, 24, 35, 36] that the Floquet operator K_F is of pure

point spectra or no absolutely continuous spectra where P is bounded. When P is unbounded, the first result was obtained by Bambusi and Graffi [1] where they considered the time dependent Schrödinger equation

$$i\partial_t\psi(x, t) = H(t)\psi(x, t), x \in \mathbb{R}; \quad H(t) := -\frac{d^2}{dx^2} + Q(x) + \epsilon V(x, \omega t), \quad \epsilon \in \mathbb{R}, \quad (1.8)$$

where $Q(x) \sim |x|^\alpha$ with $\alpha > 2$ as $|x| \rightarrow \infty$ and $|V(x, \theta)||x|^{-\beta}$ is bounded as $|x| \rightarrow \infty$ for some $\beta < \frac{\alpha-2}{2}$. This entails the pure-point nature of the spectrum of the Floquet operator

$$K_F := -i\omega \cdot \partial_\theta - \frac{d^2}{dx^2} + Q(x) + \epsilon V(x, \theta),$$

on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T}^n)$ for ϵ small. Liu and Yuan [33] solved the limit case when $\alpha > 2$ and $\beta \leq \frac{\alpha-2}{2}$, which can be applied to the so-called quantum version of the Duffing oscillator

$$i\partial_t\psi(x, t) = \left(-\frac{d^2}{dx^2} + x^4 + \epsilon x V(\omega t)\right)\psi(x, t), \quad x, \epsilon \in \mathbb{R}.$$

The results in [1] and [33] didn't include the case $Q(x) = x^2$ (see (1.8)), which is so-called quantum harmonic oscillator ($\alpha = 2$). The quantum harmonic oscillator is the quantum-mechanical analog of the classical harmonic oscillator. Because an arbitrary potential can usually be approximated as a harmonic potential at the vicinity of a stable equilibrium point, it is one of the most important model systems in quantum mechanics.

In [15] Enns and Veselic proved that, if ω is rational, the Floquet operator K (see (1.6)) has pure point spectrum when the perturbing potential V is bounded and has sufficiently fast decay at infinity. In [42] Wang proved the spectrum of the Floquet operator K is pure point where the perturbing potential $V(x, \theta) = e^{-x^2} \sum_{k=1}^n \cos \theta_k$ has *exponential decay*. Grebért and Thomann [22] improved the results in [42] from exponential decay to *polynomial decay*. In this paper we improve the results in [22] from polynomial decay to *logarithmic decay*. But we know nothing about K_F when $\alpha = 2$ and $\beta = 0$, i.e. $Q(x) \sim |x|^2$ as $|x| \rightarrow \infty$ and $V(x, \theta)$ is only bounded (except the special case when $V(x, \theta)$ is independent of x , see [22]). This problem was firstly posed by Eliasson in [10] and is still open till now.

For V is unbounded it is a different situation. See [20, 23, 43] and related discussions in [42].

1.3. The main idea for proving Theorem 1.1. As in [1], [13] and [22] the proof of Theorem 1.1 is closely related to an infinite dimensional KAM theorem. Since the formulation of this abstract theorem is technical and lengthy, we postpone it to Sect. 2, see Theorem 2.2 (KAM Theorem). Let us firstly explain the main idea and techniques for proving Theorem 2.2.

We begin with a parameter dependent family of analytic Hamiltonians of the form

$$\begin{aligned} H &= N(y, z, \bar{z}; \xi) + P(\theta, y, z, \bar{z}; \xi) \\ &= \sum_{1 \leq j \leq n} \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) z_j \bar{z}_j + P(\theta, y, z, \bar{z}; \xi), \end{aligned}$$

where $(\theta, y) \in \mathbb{T}^n \times \mathbb{R}^n$, $z = (z_j)_{j \geq 1}$, $\bar{z} = (\bar{z}_j)_{j \geq 1}$ are infinitely many variables, $\omega(\xi) = (\omega_j(\xi)) \in \mathbb{R}^n$, $\Omega(\xi) = (\Omega_j(\xi)) \in \mathbb{R}^\infty$ and $\xi \in \Pi \subset \mathbb{R}^n$. For our applications we suppose $\Omega_j(\xi) = 2j-1$ for simplicity and $\langle l, \Omega(\xi) \rangle \neq 0$, $\forall 1 \leq |l| \leq 2$. Our aim is to find a suitable real analytic symplectic coordinates transformation Φ such that $H \circ \Phi = N^* + P^*$ where N^* has a similar form as N and P^* is analytic and globally of order 3.

Actually, following the formulation of the KAM theorem given in [38] (see also [22]), it is sufficient to verify that there exist $\xi \in \Pi$ with big Lebesgue measure which satisfy

$$|\langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle| \geq \frac{\langle l \rangle \alpha_\nu}{1 + |k|^\tau}, \quad (1.9)$$

where $\omega_\nu(\xi), \Omega_\nu(\xi)$ are the frequencies in ν -th KAM step.

To obtain (1.9) in most cases we need some assumptions for X_P such as [28] and [38]. See Theorem 1 in [28] and its applications to 1d wave equation and 1d harmonic oscillator with a smooth nonlinearity of type $P = \frac{1}{2} \int_{\mathbb{R}} \varphi(|u * \xi(x)|^2; a) dx$ ($u * \xi$ is the convolution with a smooth real-valued function ξ , vanishing at infinity). In [38] Pöschel required a similar condition on X_P , i.e.

$$X_P : \mathcal{P}^p \rightarrow \mathcal{P}^{\bar{p}}, \quad \bar{p} > p. \quad (1.10)$$

See [39] for its application to a nonlinear wave equation.

The assumption (1.10) is smartly weakened in [22]. Using the Töplitz-Lipschitz techniques from Eliasson and Kuksin in [14], Grébert and Thomann assumed a weaker regularity on P , more clearly,

$$\left\| \frac{\partial P}{\partial \omega_j} \right\|_{D(s,r)}^* \leq \frac{r}{j^{\beta_1}} \langle P \rangle_{r,D(s,r)}^*, \quad \left\| \frac{\partial^2 P}{\partial \omega_j \partial \omega_l} \right\|_{D(s,r)}^* \leq \frac{1}{(jl)^{\beta_1}} \langle P \rangle_{r,D(s,r)}^*,$$

for all $j, l \geq 1$ and $\omega_j = z_j, \bar{z}_j$, where $\beta_1 > 0$ and $\|\cdot\|_{D(s,r)}^*$ stands for either $\|\cdot\|_{D(s,r)}$ or $\|\cdot\|_{D(s,r)}^S$. To recover this assumption at each step they noticed that F satisfied an even better estimate, i.e.

$$\left\| \frac{\partial F}{\partial \omega_j} \right\|_{D(s,r)}^* \leq \frac{r}{j^{\beta_1+1}} \langle F \rangle_{r,D(s,r)}^{+,*}, \quad \left\| \frac{\partial^2 F}{\partial \omega_j \partial \omega_l} \right\|_{D(s,r)}^* \leq \frac{\langle F \rangle_{r,D(s,r)}^{+,*}}{(jl)^{\beta_1} (1 + |j-l|)}$$

for all $j, l \geq 1$ and $\omega_j = z_j, \bar{z}_j$.

In this paper we further weaken the assumptions on P , which satisfies the logarithmic decay, i.e.

$$\left\| \frac{\partial P}{\partial \omega_j} \right\|_{D(s,r)}^* \leq \frac{r}{(1 + \ln j)^\beta} \langle P \rangle_{r,D(s,r)}^*, \quad \left\| \frac{\partial^2 P}{\partial \omega_j \partial \omega_l} \right\|_{D(s,r)}^* \leq \frac{\langle P \rangle_{r,D(s,r)}^*}{(1 + \ln j)^\beta (1 + \ln l)^\beta}$$

for all $j, l \geq 1$ and $\omega_j = z_j, \bar{z}_j$. The index $\beta \geq 2(n+2)$ is apparently different from $\beta_1 > 0$ in [22], which we will explain in the following. As in [22] we obtain a better estimate for F , i.e.

$$\left\| \frac{\partial F}{\partial \omega_j} \right\|_{D(s,r)}^* \leq \frac{r}{j(1 + \ln j)^\beta} \langle F \rangle_{r,D(s,r)}^{+,*}, \quad \left\| \frac{\partial^2 F}{\partial \omega_j \partial \omega_l} \right\|_{D(s,r)}^* \leq \frac{\langle F \rangle_{r,D(s,r)}^{+,*}}{(1 + \ln j)^\beta (1 + \ln l)^\beta (1 + |j-l|)}$$

for all $j, l \geq 1$ and $\omega_j = z_j, \bar{z}_j$.

The shift in normal frequencies in the next step $\Omega_j^+(\xi) = \Omega_j(\xi) + \widehat{\Omega}_j(\xi)$ satisfies much weaker estimate

$$|\widehat{\Omega}_j| \preceq \alpha(1 + \ln j)^{-2\beta}, \quad j \geq 1, \quad (1.11)$$

comparing with the corresponding one in [22], which is

$$|\widehat{\Omega}_j| \preceq \alpha j^{-2\beta}, \quad j \geq 1. \quad (1.12)$$

Fact proved that the weaker estimate (1.11) brought new troubles in measure estimates. The consequence is that we need a new small divisor condition

$$|\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| \geq \frac{\langle l \rangle \alpha}{\exp(|k|^{\tau/\beta})}, \quad \beta \geq 2\tau \geq 2(n+2), \quad (1.13)$$

in the KAM proof, which will be clear in the following.

For simplicity we suppose $\omega(\xi) = \xi$ and $\Omega_j(\xi) = 2j - 1 + O(\alpha(1 + \ln j)^{-2\beta})$ ($j \geq 1$). Our main trouble is to estimate the set

$$\bigcup_{\substack{k,j,b \\ 1 \leq b \leq c|k|}} \left\{ \xi \in \Pi : |f_{k,jb}(\xi)| < \frac{\alpha b}{\Delta_1(|k|)} \right\},$$

where

$$f_{k,jb}(\xi) := \langle k, \omega(\xi) \rangle + \Omega_i(\xi) - \Omega_j(\xi) = \langle k, \xi \rangle + 2b + O(\alpha(1 + \ln j)^{-2\beta}),$$

$i = j + b$ and $\Delta_1(\cdot)$ will be chosen later. From a straightforward computation (see 1.11), if $j \geq j_0 = \exp\{(c\Delta(|k|))^{\frac{1}{2\beta}}\}$, then $|f_{k,jb}(\xi)| \geq \alpha b \Delta^{-1}(|k|) \geq \alpha b \Delta_1^{-1}(|k|)$, where $\Delta_1(|k|) \geq \Delta(|k|)$ and $|\langle k, \xi \rangle + 2b| \geq 2\alpha b \Delta^{-1}(|k|)$. For the rest,

$$Meas\left(\bigcup_{\substack{1 \leq j \leq j_0 \\ 1 \leq b \leq c|k|}} \{\xi \in \Pi : |f_{k,jb}(\xi)| < \frac{\alpha b}{\Delta_1(|k|)}\}\right) \leq \frac{\alpha |k|^2}{\Delta_1(|k|)} \cdot \exp\{(c\Delta(|k|))^{\frac{1}{2\beta}}\}. \quad (1.14)$$

(1.14) explains (1.13) since we set $\Delta(l) = l^\tau$, and thus $\Delta_1(l) = \exp(l^{\tau/\beta})$. The index $\beta \geq 2(n+2)$ comes from Lemma 5.1 ($\beta > \tau$) and $\beta = \iota\tau \geq 2\tau \geq 2(n+2)$ (see Theorem 6.1 for $\tau \geq n+2$). In the KAM proof we set $\iota \geq 2$ for simplicity.

We remark that the structure (1.11) or (1.12) is not necessary for some evolution equations. Based on the work [14], Berti, Biasco and Procesi in [3, 4] found a remarkable structure for $\Omega_j^+(\xi) = \Omega_j(\xi) + \widehat{\Omega}_j(\xi)$ in 1d derivative wave equation where $\widehat{\Omega}_j(\xi) = a_+(\xi) + O(\frac{1}{j})$ ($j \gg 1$) and $a_+(\xi)$ is independent of j (similar for $j < 0$). But we don't know whether there exists similar or weaker structure for 1d harmonic oscillator, which is a potential way to solve Eliasson's problem.

To finish the proof of Theorem 1.1 we will follow the scheme given by Eliasson and Kuksin in [13]. As in [13] the equation (1.1) is rewritten into an autonomous Hamiltonian system in an extended phase space $\mathcal{P}^2 := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{2,2} \times \ell^{2,2}$, with the Hamiltonian function $H = N + \varepsilon P$, where

$$N := N(\omega) = \sum_{1 \leq j \leq n} \omega_j y_j + \sum_{j \geq 1} (2j-1) z_j \bar{z}_j.$$

and

$$P(\theta, z, \bar{z}) = \int_{\mathbb{R}} V(x, \theta; \omega) \left(\sum_{j \geq 1} z_j h_j \right) \left(\sum_{j \geq 1} \bar{z}_j h_j \right) dx$$

is quadratic in (z, \bar{z}) . Here the external parameters are the frequencies $\omega = (\omega_j)_{1 \leq j \leq n} \in \Pi := [0, 2\pi]^n$ and the normal frequencies $\Omega_j = 2j - 1$ are independent of ω . To apply Theorem 2.2 into the above Hamiltonian we need to check all the assumptions in Theorem 2.2 are satisfied, and thus finish the proof of Theorem 1.1. In fact we need an improved estimate on $h_n(x)$. For simplicity we define the weighted L^2 norm of $h_n(x)$,

$$|||h_n(x)||| = \left(\int_{\mathbb{R}} \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{2\delta_1}} dx \right)^{\frac{1}{2}},$$

with $\delta_1 > 0$.

Lemma 1.4. *Suppose $h_n(x)$ satisfies (1.5), then for any $n \geq 1$,*

$$|||h_n(x)||| \leq \frac{C_{\delta_1}}{(1 + \ln n)^{\delta_1}},$$

where C_{δ_1} is a constant depending on δ_1 only.

In the end of this section we give a fast description on the recent development in KAM theory. For the KAM results with bounded perturbations see [17, 25, 27, 28, 31, 32] for 1d-NLS. For high dimensional NLS see the milestone work by Eliasson and Kuksin [14], where they found and defined a Töplitz-Lipschitz property and used it to control the shift of the normal frequencies. See [16] and [37] for recent development in nd-NLS. For nd-beam equations see [18] and [19] for the nonlinearity $g(u)$ and see [11] and [12] for more general nonlinearity $g(x, u)$. Adapting the technics in [14] and [22], Grébert and Paturel built a KAM for the Klein Gordon equation on S^d in [21]. We remark that an earlier results for NLW and NLS on compact Lie groups via Nash Moser technics can be found in [5] (see also [6]), in which Berti, Corsi and Procesi proved the existence of quasi-periodic solutions without linear stability.

For the unbounded perturbations, the first KAM results have been obtained by Kuksin [29]-[30]

for KdV with analytic perturbations(also see Kappeler and Pöschel [26]). See [34] and [45] for recent progress for 1d derivative nonlinear Schrödinger equation(DNLS), Benjamin-Ono equation and the reversible DNLS equation.

Plan of the proof of Theorem 1.1. In Sect. 2 we give an abstract KAM theorem(Thm. 2.2) which will be used to prove Thm. 4.1, a Hamiltonian formulation of Thm. 1.1, in Sect. 4. In Sect. 3 we prove the weighted L^2 estimates on the Hermite basis. The KAM proof of Thm. 2.2 is deferred in Sect. 5 while the measure estimates are presented in Sect. 6. We put two technical inequalities in the Appendix.

Notations. $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^d or \mathbb{R}^∞ . $\|\cdot\|$ is an operator-norm or ℓ^2 -norm. $|\cdot|$ will in general denote a supremum norm with a notable exception. For $l = (l_1, l_2, \dots, l_k, \dots) \in \mathbb{Z}^\infty$ so that only a finite number of coordinates are nonzero, we denote by $|l| = \sum_{j=1}^\infty |l_j|$ its length, $\langle l \rangle = \max\{1, |\sum_{j=1}^\infty j l_j|\}$. We use the notations $\mathbb{Z}_+ = \{1, 2, \dots\}$ and $\mathbb{N} = \{0, 1, 2, \dots\}$.

The notation “ \leq ” used below means \leq modulo a multiplicative constant that, unless otherwise specified, depends on n only.

We set $\mathcal{Z} = \{(k, l) \neq 0, |l| \leq 2\} \subset \mathbb{Z}^n \times \mathbb{Z}^\infty$. Denote by $\Delta_{\xi\eta}$ the difference operator in the variable ξ , $\Delta_{\xi\eta} f = f(\cdot, \xi) - f(\cdot, \eta)$, where f is a real function.

We denote $\ell_{\mathbb{C}}^{2,p}$ the Hilbert space of all complex sequences $w = (w_j)_{j \geq 1}$ with $\|w\|_p^2 = \sum_{j \geq 1} j^p |w_j|^2 < \infty$. We denote ℓ_∞^q the space of all real(complex) sequences with finite norm $|w|_{\ell_\infty^q} = \sup_{j \geq 1} |w_j| j^q < \infty$, where $q \in \mathbb{R}$.

Following [38], we use $\|\cdot\|^*$ (respectively $\langle \cdot \rangle^*$) stands either for $\|\cdot\|$ or $\|\cdot\|^\mathcal{L}$ (respectively $\langle \cdot \rangle$ or $\langle \cdot \rangle^\mathcal{L}$) and $\|\cdot\|^\lambda$ stands for $\|\cdot\| + \lambda \|\cdot\|^\mathcal{L}$. The notation Meas stands for the Lebesgue measure in \mathbb{R}^n .

2. KAM THEOREM

2.1. KAM theorem. Following the exposition in [22], [25] and [38], we consider small perturbations of a family of infinite-dimensional integrable Hamiltonians $N(y, u, v; \xi)$ with parameter ξ in the normal form

$$N = \sum_{1 \leq j \leq n} \omega_j(\xi) y_j + \frac{1}{2} \sum_{j \geq 1} \Omega_j(\xi) (u_j^2 + v_j^2), \quad (2.1)$$

on the phase space $\mathcal{M}^p := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{2,p} \times \ell^{2,p}$ with coordinates (θ, y, u, v) . The ‘internal’ frequencies, $\omega = (\omega_j)_{1 \leq j \leq n}$, as well as the ‘external’ ones, $\Omega = (\Omega_j)_{j \geq 1}$, are real valued and depend on the parameter $\xi \in \Pi \subset \mathbb{R}^n$ and Π is a compact subset of \mathbb{R}^n of positive Lebesgue measure. The symplectic structure on \mathcal{M}^p is the standard one given by $\sum_{1 \leq j \leq n} d\theta_j \wedge dy_j + \sum_{j \geq 1} du_j \wedge dv_j$.

The Hamiltonian equations for motion of N are therefore

$$\dot{\theta} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{u} = \Omega(\xi)v, \quad \dot{v} = -\Omega(\xi)u,$$

where for any $j \geq 1$, $(\Omega(\xi)u)_j = \Omega_j u_j$. Hence, for any parameter $\xi \in \Pi$, on the n -dimensional invariant torus,

$$\mathbb{T}_0 = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\},$$

the flow is rotational with internal frequencies $\omega(\xi)$. In the normal space, described by the (u, v) coordinates, we have an elliptic equilibrium at the origin, whose frequencies are $\Omega(\xi) = (\Omega_j)_{j \geq 1}$. Hence, for any $\xi \in \Pi$, \mathbb{T}_0 is an invariant, rotational, linearly stable torus for the Hamiltonian N .

Our aim is to prove the persistence of this torus under small perturbations $N+P$ of the integrable Hamiltonian N for a large Cantor set of parameter values ξ . To this end we make assumptions on the frequencies of the unperturbed Hamiltonian N and on the perturbation P .

We need some notations for simplification. In the sequel, we use the distance

$$\|\Omega - \Omega'\|_{2\beta, \Pi} = \sup_{\xi \in \Pi} \sup_{j \geq 1} (1 + \ln j)^{2\beta} |\Omega_j(\xi) - \Omega'_j(\xi)|,$$

and the semi-norm,

$$\|\Omega\|_{2\beta, \Pi}^{\mathfrak{L}} = \sup_{\substack{\xi, \eta \in \Pi \\ \xi \neq \eta}} \sup_{j \geq 1} \frac{(1 + \ln j)^{2\beta} |\Delta_{\xi\eta} \Omega_j|}{|\xi - \eta|}.$$

Assumption \mathcal{A} (Frequencies):

(A1) The map $\xi \mapsto \omega(\xi)$ between Π and its image $\omega(\Pi)$ is a homeomorphism which, together with its inverse, is Lipschitz continuous.

(A2) There exists a real sequence $(\bar{\Omega}_j)_{j \geq 1}$, independent of $\xi \in \Pi$, of the form $\bar{\Omega}_j = a_1 j + a_2$ with $a_1, a_2 \in \mathbb{R}$ and $a_1 \neq 0$, so that $\xi \mapsto (\Omega_j - \bar{\Omega}_j)_{j \geq 1}$ is a Lipschitz continuous map on Π with values in $\ell_{\infty}^{-\delta}(\delta < 0)$. More clearly, for $\xi \in \Pi$, $|\Omega - \bar{\Omega}|_{\ell_{\infty}^{-\delta}} \leq M_1$ with $M_1 > 0$.

(A3) For all $(k, l) \in \mathcal{Z}$,

$$\text{Meas}(\{\xi : k \cdot \omega(\xi) + l \cdot \Omega(\xi) = 0\}) = 0.$$

and for all $\xi \in \Pi$,

$$l \cdot \Omega(\xi) \neq 0, \quad \forall 1 \leq |l| \leq 2.$$

The second set of assumptions concerns the perturbing Hamiltonian P and its vector field, $X_P = (\partial_y P, -\partial_{\theta} P, \partial_v P, -\partial_u P)$. We use the notation $i_{\xi} X_P$ for X_P evaluated at ξ . Finally, we denote by $\mathcal{M}_{\mathbb{C}}^p$ the complexification of the phase space \mathcal{M}^p , $\mathcal{M}_{\mathbb{C}}^p = (\mathbb{C}/2\pi\mathbb{Z})^n \times \mathbb{C}^n \times \ell_{\mathbb{C}}^{2,p} \times \ell_{\mathbb{C}}^{2,p}$. Note that at each point of $\mathcal{M}_{\mathbb{C}}^p$, the tangent space is given by

$$\mathcal{P}_{\mathbb{C}}^p := \mathbb{C}^n \times \mathbb{C}^n \times \ell_{\mathbb{C}}^{2,p} \times \ell_{\mathbb{C}}^{2,p}.$$

To state the assumptions about the perturbation we need to introduce some domains and norms. For $s, r > 0$, we define the complex neighborhood of \mathbb{T}_0 -neighborhoods

$$D(s, r) = \{|\Im \theta| < s\} \times \{|y| < r^2\} \times \{\|u\|_p + \|v\|_p < r\} \subset \mathcal{M}_{\mathbb{C}}^p.$$

Here, for a in \mathbb{R}^n or \mathbb{C}^n , $|a| = \max_j |a_j|$ and $p \geq 2$. Let $r > 0$, then for $W = (X, Y, U, V)$ in $\mathcal{P}_{\mathbb{C}}^p$ we denote

$$|W|_r = |X| + \frac{1}{r^2} |Y| + \frac{1}{r} (\|U\|_p + \|V\|_p).$$

We then define the norms

$$\|P\|_{D(s, r)} := \sup_{D(s, r) \times \Pi} |P|, \quad \|P\|_{D(s, r)}^{\mathfrak{L}} := \sup_{\substack{\xi, \eta \in \Pi \\ \xi \neq \eta}} \sup_{D(s, r)} \frac{|\Delta_{\xi\eta} P|}{|\xi - \eta|},$$

and we define the semi-norms

$$\|X_P\|_{r, D(s, r)} := \sup_{D(s, r) \times \Pi} |X_P|_r, \quad \|X_P\|_{r, D(s, r)}^{\mathfrak{L}} := \sup_{\substack{\xi, \eta \in \Pi \\ \xi \neq \eta}} \sup_{D(s, r)} \frac{|\Delta_{\xi\eta} X_P|_r}{|\xi - \eta|}.$$

In the sequel, we will often work in the complex coordinates $z = \frac{1}{\sqrt{2}}(u - iv)$, $\bar{z} = \frac{1}{\sqrt{2}}(u + iv)$. Notice that this is not a canonical change of variables and in the variables $(\theta, y, z, \bar{z}) \in \mathcal{M}_{\mathbb{C}}^p$ the symplectic structure reads $\sum_{1 \leq j \leq n} d\theta_j \wedge dy_j + i \sum_{j \geq 1} dz_j \wedge d\bar{z}_j$, and the Hamiltonian in normal form is

$$N = \sum_{1 \leq j \leq n} \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) z_j \bar{z}_j, \quad (2.2)$$

Assumption \mathcal{B} (Perturbation):

(B1) We assume that there exist $s, r > 0$ so that

$$X_P : D(s, r) \times \Pi \longrightarrow \mathcal{P}_{\mathbb{C}}^p.$$

Moreover $i_{\xi}X_P(\cdot, \xi)$ is analytic in $D(s, r)$ for each $\xi \in \Pi$. $i_w P$ and $i_w X_P$ are uniformly Lipschitz on Π for each $w \in D(s, r)$.

Similar as [22], we denote $\Gamma_{r, D(s, r)}^{\beta}$ as the following: Let $\beta > 0$, we say that $P \in \Gamma_{r, D(s, r)}^{\beta}$ if $\langle P \rangle_{r, D(s, r)} + \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}} < \infty$ where the norm $\langle \cdot \rangle_{r, D(s, r)}$ is defined by the conditions

$$\begin{aligned} \|P\|_{D(s, r)} &\leq r^2 \langle P \rangle_{r, D(s, r)}, \\ \max_{1 \leq j \leq n} \left\| \frac{\partial P}{\partial y_j} \right\|_{D(s, r)} &\leq \langle P \rangle_{r, D(s, r)}, \\ \left\| \frac{\partial P}{\partial \omega_j} \right\|_{D(s, r)} &\leq \frac{r}{(1 + \ln j)^{\beta}} \langle P \rangle_{r, D(s, r)}, \quad \forall j \geq 1 \quad \text{and} \quad \omega_j = z_j, \bar{z}_j, \\ \left\| \frac{\partial^2 P}{\partial \omega_j \partial \omega_l} \right\|_{D(s, r)} &\leq \frac{1}{(1 + \ln j)^{\beta} (1 + \ln l)^{\beta}} \langle P \rangle_{r, D(s, r)}, \quad \forall j, l \geq 1 \quad \text{and} \quad \omega_j = z_j, \bar{z}_j, \end{aligned}$$

and the semi-norm $\langle \cdot \rangle_{r, D(s, r)}^{\mathfrak{L}}$ is defined by the conditions

$$\begin{aligned} \|P\|_{D(s, r)}^{\mathfrak{L}} &\leq r^2 \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}}, \\ \max_{1 \leq j \leq n} \left\| \frac{\partial P}{\partial y_j} \right\|_{D(s, r)}^{\mathfrak{L}} &\leq \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}}, \\ \left\| \frac{\partial P}{\partial \omega_j} \right\|_{D(s, r)}^{\mathfrak{L}} &\leq \frac{r}{(1 + \ln j)^{\beta}} \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}}, \quad \forall j \geq 1 \quad \text{and} \quad \omega_j = z_j, \bar{z}_j, \\ \left\| \frac{\partial^2 P}{\partial \omega_j \partial \omega_l} \right\|_{D(s, r)}^{\mathfrak{L}} &\leq \frac{1}{(1 + \ln j)^{\beta} (1 + \ln l)^{\beta}} \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}}, \quad \forall j, l \geq 1 \quad \text{and} \quad \omega_j = z_j, \bar{z}_j. \end{aligned}$$

(B2) $P \in \Gamma_{r, D(s, r)}^{\beta}$ for some $\beta = \iota\tau \geq \iota(n+2)$ where $\iota \geq 2$.

Remark 2.1. In the application to 1d quantum harmonic oscillator we will choose $\beta \geq 2(n+2)$, which is NOT the best choice. But we have no intent to obtain the optimal one for β .

We set $|\omega|_{\Pi} \leq M$, $|\omega^{-1}|_{\Pi}^{\mathfrak{L}} \leq L$ and

$$|\omega|_{\Pi}^{\mathfrak{L}} + \|\Omega\|_{2\beta, \Pi}^{\mathfrak{L}} \leq M, \quad (2.3)$$

where $|\omega|_{\Pi}^{\mathfrak{L}} = \sup_{\xi, \eta \in \Pi} \max_{\substack{1 \leq k \leq n \\ \xi \neq \eta}} \frac{|\Delta_{\xi\eta}\omega_k|}{|\xi - \eta|}$.

Theorem 2.2. (KAM) Suppose that N is a family of Hamiltonian of the form (2.2) defined on the phase space \mathcal{M}^p with $p \geq 2$ depending on parameters $\xi \in \Pi$ so that Assumption \mathcal{A} is satisfied. Then there exist $\gamma > 0$ and $s > 0$ so that every perturbation $H = N + P$ of N which satisfies Assumption \mathcal{B} and the smallness condition

$$\varepsilon = (\|X_P\|_{r, D(s, r)} + \langle P \rangle_{r, D(s, r)}) + \frac{\alpha}{M} (\|X_P\|_{r, D(s, r)}^{\mathfrak{L}} + \langle P \rangle_{r, D(s, r)}^{\mathfrak{L}}) \leq \gamma\alpha^5,$$

for some $r > 0$ and $0 < \alpha \leq 1$, the following holds.

There exist

- (i) a Cantor set $\Pi_{\alpha} \subset \Pi$ with $\text{Meas}(\Pi \setminus \Pi_{\alpha}) \mapsto 0$ as $\alpha \mapsto 0$;
- (ii) a Lipschitz family of real analytic, symplectic coordinates transformations

$$\Phi : D(s/2, r/2) \times \Pi_{\alpha} \longrightarrow D(s, r);$$

(iii) a Lipschitz family of new normal form

$$N^* = \sum_{j=1}^n \omega_j^*(\xi) y_j + \sum_{j \geq 1} \Omega_j^*(\xi) z_j \bar{z}_j$$

defined on $D(s/2, r/2) \times \Pi_\alpha$ such that

$$H \circ \Phi = N^* + P^*,$$

where P^* is analytic on $D(s/2, r/2)$ and globally of order 3 at \mathbb{T}_0 . That is the Taylor expansion of P^* only contains monomials $y^m z^q \bar{z}^{\bar{q}}$ with $2|m| + |q + \bar{q}| \geq 3$. Moreover each symplectic coordinates transformation is close to the identity

$$\|\Phi - Id\|_{r, D(s/2, r/2)} \leq c\varepsilon^{1/2}, \quad (2.4)$$

and the new frequencies are close to the original ones

$$|\omega^* - \omega|_{\Pi_\alpha} + \|\Omega^* - \Omega\|_{2\beta, \Pi_\alpha} \leq c\varepsilon, \quad (2.5)$$

and the new frequencies satisfy a non resonance condition

$$|k \cdot \omega^*(\xi) + l \cdot \Omega^*(\xi)| \geq \frac{\alpha}{2} \cdot \frac{\langle l \rangle}{\exp(|k|^{1/\iota})}, \quad \iota \geq 2, \quad (k, l) \in \mathcal{Z}, \quad \xi \in \Pi_\alpha. \quad (2.6)$$

Remark 2.3. As a consequence, for each $\xi \in \Pi_\alpha$ the torus $\Phi(\mathbb{T}_0)$ is invariant under the flow of the perturbed Hamiltonian $H = N + P$ and all these tori are linearly stable.

3. ESTIMATES ON EIGENFUNCTIONS IN A WEIGHTED L^2 NORM

In this section we will prove Lemma 1.4. A well-known fact is that $h_n(x) = (n!2^n \pi^{\frac{1}{2}})^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} H_n(x)$ where $H_n(x)$ is the Hermite polynomial of degree n and $h_n(x)$ is an even or odd function of x according to whether n is odd or even (see Titchmarsh [40]). The proof of Lemma 1.4 is based upon Langer's turning point theory as presented in Chapter 22.27 of [41] (see [44]). For simplicity we define the weighted L^2 norms of $h_n(x)$ on \mathbb{R} and \mathbb{R}_\pm , which are

$$|||h_n(x)||| = \left(\int_{\mathbb{R}} \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{2\delta_1}} dx \right)^{\frac{1}{2}},$$

and

$$|||h_n(x)|||_\pm = \left(\int_{\mathbb{R}_\pm} \frac{h_n^2(x)}{(1 + \ln(1 + x^2))^{2\delta_1}} dx \right)^{\frac{1}{2}}$$

with $\delta_1 > 0$.

From the symmetry $|||h_n(x)|||^2 = 2|||h_n(x)|||_+^2$, and thus we only need to estimate $|||h_n(x)|||_+$. In the following we assume n be sufficiently large. As in [44],

$$\begin{aligned} h_n(x) &= (\lambda_n - x^2)^{-\frac{1}{4}} \left(\frac{\pi\zeta}{2} \right)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta) + (\lambda_n - x^2)^{-\frac{1}{4}} \left(\frac{\pi\zeta}{2} \right)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta) \mathcal{O}\left(\frac{1}{\lambda_n}\right) \\ &:= \psi_1(x) + \psi_2(x), \end{aligned}$$

where $\zeta(x) = \int_X^x \sqrt{\lambda_n - t^2} dt$ with $X^2 = \lambda_n$. We only need to estimate $\psi_1(x)$ since the estimate for $\psi_2(x)$ is even better. Let

$$Q(y) = \begin{cases} -\int_1^y \sqrt{1-s^2} ds, & \text{if } y < 1, \\ i \int_1^y \sqrt{s^2-1} ds, & \text{if } y > 1. \end{cases}$$

We have $\zeta(x) = \lambda_n Q(y)$. By Lemma 2.2 in [44] it holds that for any $K > 1$

$$\begin{aligned} Q(y) &\sim -(1-y)^{\frac{3}{2}}, & \text{for } 0 \leq y \leq 1, \\ -iQ(y) &\sim (y-1)^{\frac{3}{2}}, & \text{for } 1 \leq y \leq K, \\ -iQ(y) &\sim y^2, & \text{for } y \geq K. \end{aligned}$$

Recall that $H_{\frac{1}{3}}^{(1)}(\zeta)$ satisfies the following ([40]):

(1) when $\zeta = -z < 0$, $H_{\frac{1}{3}}^{(1)}(\zeta) = \frac{2}{\sqrt{3}}e^{-\frac{1}{6}\pi i}(J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z))$ and

$$\zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta) = \begin{cases} 2^{\frac{2}{3}}\pi^{-\frac{1}{2}}e^{\frac{1}{3}\pi i}(\cos(z - \pi/4) + \mathcal{O}(z^{-1})), & z \rightarrow \infty, \\ 2^{\frac{2}{3}}3^{-\frac{1}{2}}\Gamma(2/3)^{-1}e^{\frac{1}{3}\pi i}z^{\frac{1}{6}}(1 + \mathcal{O}(z)), & z \rightarrow 0, \end{cases} \quad (3.1)$$

(2) when $\zeta = iw$ and $w \geq 0$, $H_{\frac{1}{3}}^{(1)}(\zeta) = \frac{2}{\pi}e^{-\frac{2}{3}\pi i}K_{\frac{1}{3}}(w)$ and

$$\zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta) = \begin{cases} \mathcal{O}(e^{-w}), & w \rightarrow \infty, \\ 2^{\frac{1}{3}}e^{-\frac{1}{6}\pi i}\pi^{-1}\Gamma(1/3)w^{\frac{1}{6}} + \mathcal{O}(w^{\frac{3}{2}}), & w \rightarrow 0. \end{cases} \quad (3.2)$$

If n is large, then

$$\begin{aligned} |||h_n|||_+^2 &\leq 2 \int_0^{+\infty} \frac{|\psi_1(x)|^2}{(1 + \ln(1 + x^2))^{2\delta_1}} dx + 2 \int_0^{+\infty} \frac{|\psi_2(x)|^2}{(1 + \ln(1 + x^2))^{2\delta_1}} dx \\ &\leq C \int_0^{+\infty} \frac{|\zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)|^2}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy. \end{aligned}$$

Lemma 1.4 is a direct corollary from the following lemma.

Lemma 3.1. *There exists $C > 0$ such that for large n ,*

$$\int_0^{+\infty} \frac{|\zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)|^2}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy \leq \frac{C \cdot 2^{2\delta_1}}{(1 + \ln n)^{2\delta_1}}. \quad (3.3)$$

Proof. We split the integral into three parts

$$\left(\int_0^1 + \int_1^K + \int_K^{+\infty} \right) \frac{|\zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)|^2}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy = I_1 + I_2 + I_3$$

and estimate them separately.

(1) When $0 \leq y \leq 1$, $\zeta = -z < 0$, we split the integral I_1 into two parts $I_1 = I_{11} + I_{12}$. Applying the first relation of (3.1) to I_{11} and the second to I_{12} ,

$$\begin{aligned} I_{11} &\leq \int_0^{1-X^{-\frac{2}{3}}} \frac{C}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy \\ &\leq \left(\int_0^{X^{-\frac{1}{2}}} + \int_{X^{-\frac{1}{2}}}^1 \right) \frac{C}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy \\ &\leq C \left(X^{-\frac{1}{2}} + \frac{1}{(1 + \ln(1 + X))^{2\delta_1}} \right) \leq \frac{C \cdot 2^{2\delta_1}}{(\ln n)^{2\delta_1}}, \end{aligned}$$

and

$$\begin{aligned} I_{12} &\leq C \int_{1-X^{-\frac{2}{3}}}^1 \frac{\zeta^{\frac{1}{3}}}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy \\ &\leq \int_{1-X^{-\frac{2}{3}}}^1 \frac{CX^{\frac{2}{3}}(1 - y)^{\frac{1}{2}}}{|1 - y^2|^{\frac{1}{2}}(1 + \ln(1 + y^2X^2))^{2\delta_1}} dy \leq \frac{C \cdot 2^{2\delta_1}}{(\ln n)^{2\delta_1}}. \end{aligned}$$

(2) When $1 \leq y \leq K$ and $w = -i\zeta \geq 0$, we split the integral I_2 into two parts $I_2 = I_{21} + I_{22}$. Applying the first relation of (3.2) to I_{21} and the second to I_{22} , we obtain

$$\begin{aligned} I_{21} &\leq \int_{1+X^{-\frac{2}{3}}}^K \frac{Ce^{-2X^2(y-1)^{\frac{3}{2}}}}{|1-y^2|^{\frac{1}{2}}(1+\ln(1+y^2X^2))^{2\delta_1}} dy \\ &\leq \frac{C}{(\ln n)^{2\delta_1}} \int_{1+X^{-\frac{2}{3}}}^K \frac{e^{-2X^2(y-1)^{\frac{3}{2}}}}{(y-1)^{\frac{1}{2}}} dy \leq \frac{C\lambda_n^{-\frac{2}{3}}}{(\ln n)^{2\delta_1}} \leq \frac{C}{(\ln n)^{2\delta_1}}, \end{aligned}$$

and

$$\begin{aligned} I_{22} &\leq \int_1^{1+X^{-\frac{2}{3}}} \frac{CX^{\frac{2}{3}}(y-1)^{\frac{1}{2}}}{|1-y^2|^{\frac{1}{2}}(1+\ln(1+y^2X^2))^{2\delta_1}} dy \\ &\leq \frac{C}{(1+\ln(1+X^2))^{2\delta_1}} \leq \frac{C}{(\ln n)^{2\delta_1}}. \end{aligned}$$

(3) When $y \geq K$ and $w = -i\zeta > 0$, we apply the first relation of (3.2) to I_3 ,

$$\begin{aligned} I_3 &\leq \int_K^\infty \frac{Ce^{-2w}}{|1-y^2|^{\frac{1}{2}}(1+\ln(1+y^2X^2))^{2\delta_1}} dy \\ &\leq \int_K^\infty \frac{Ce^{-2X^2y^2}}{y(1+\ln(1+y^2X^2))^{2\delta_1}} dy \\ &\leq C \frac{1}{(1+\ln(1+X^2))^{2\delta_1}} \leq \frac{C}{(1+\ln n)^{2\delta_1}}. \end{aligned}$$

Combining the above estimates together, we obtain (3.3). \square

4. APPLICATION TO QUANTUM HARMONIC OSCILLATORS

In this section, we will apply Theorem 2.2 to our model equation (1.1) and prove the results stated in Section 1. For readers' convenience, we rewrite the equation

$$i\partial_t u = -\partial_x^2 u + x^2 u + \varepsilon V(x, \omega t; \omega) u, \quad u = u(t, x), \quad x \in \mathbb{R}, \quad (4.1)$$

where the potential $V : \mathbb{R} \times \mathbb{T}^n \times \Pi \ni (x, \theta; \omega) \mapsto \mathbb{R}$ is C^3 smooth in all its variables and analytic in θ . For $\rho > 0$ the function $V(x, \theta; \omega)$ analytically in θ extends to the domain \mathbb{T}_ρ^n as well as its gradient in ω and satisfies (1.2)–(1.4) with $\beta \geq 2(n+2)$.

In the following we will follow the scheme developed by Eliasson and Kuksin in [13]. Expand u and \bar{u} on the Hermite basis $\{h_j\}_{j \geq 1}$, $u = \sum_{j \geq 1} z_j h_j$, $\bar{u} = \sum_{j \geq 1} \bar{z}_j h_j$. And thus (4.1) can be written as a nonautonomous Hamiltonian system

$$\begin{cases} \dot{z}_j = -i(2j-1)z_j - i\varepsilon \frac{\partial}{\partial \bar{z}_j} \tilde{P}(t, z, \bar{z}), & j \geq 1, \\ \dot{\bar{z}}_j = i(2j-1)\bar{z}_j + i\varepsilon \frac{\partial}{\partial z_j} \tilde{P}(t, z, \bar{z}), & j \geq 1, \end{cases} \quad (4.2)$$

where

$$\tilde{P}(t, z, \bar{z}) = \int_{\mathbb{R}} V(x, \omega t; \omega) \left(\sum_{j \geq 1} z_j h_j \right) \left(\sum_{j \geq 1} \bar{z}_j h_j \right) dx,$$

and $(z, \bar{z}) \in \ell^{2,2} \times \ell^{2,2}$. As [13] and [22], we write (4.2) as an autonomous Hamiltonian system in an extended phase space $\mathcal{P}^2 := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{2,2} \times \ell^{2,2}$,

$$\begin{cases} \dot{z}_j = -i(2j-1)z_j - i\varepsilon \frac{\partial}{\partial \bar{z}_j} P(\theta, z, \bar{z}), & j \geq 1, \\ \dot{\bar{z}}_j = i(2j-1)\bar{z}_j + i\varepsilon \frac{\partial}{\partial z_j} P(\theta, z, \bar{z}), & j \geq 1, \\ \dot{\theta}_j = \omega_j, & j = 1, 2, \dots, n, \\ \dot{y}_j = -\varepsilon \frac{\partial}{\partial \theta_j} P(\theta, z, \bar{z}), & j = 1, 2, \dots, n, \end{cases} \quad (4.3)$$

with the Hamiltonian function $H = N + \varepsilon P$, where

$$N := N(\omega) = \sum_{1 \leq j \leq n} \omega_j y_j + \sum_{j \geq 1} (2j-1) z_j \bar{z}_j.$$

and

$$P(\theta, z, \bar{z}) = \int_{\mathbb{R}} V(x, \theta; \omega) \left(\sum_{j \geq 1} z_j h_j \right) \left(\sum_{j \geq 1} \bar{z}_j h_j \right) dx$$

is quadratic in (z, \bar{z}) . Here the external parameters are the frequencies $\omega = (\omega_j)_{1 \leq j \leq n} \in \Pi := [0, 2\pi)^n$ and the normal frequencies $\Omega_j = 2j - 1$ are independent of ω . We remark that the first three equations of (4.3) are independent of y and equivalent to (4.2).

Similar as [22], we have

Theorem 4.1. *Assume that V satisfies all the conditions in Theorem 1.1 and $\beta \geq 2(n+2)$. Then there exists ε_0 such that for all $0 \leq \varepsilon < \varepsilon_0$ there exist*

- (i) $\Pi_\varepsilon \subset [0, 2\pi)^n$ of positive measure and $\text{Meas}(\Pi_\varepsilon) \rightarrow (2\pi)^n$ as $\varepsilon \rightarrow 0$;
- (ii) a Lipschitz family of real analytic, symplectic and linear coordinates transformation $\Phi : \Pi_\varepsilon \times \mathcal{P}^0 \mapsto \mathcal{P}^0$ of the form

$$\Phi_\omega(y, \theta, \zeta) = \left(y + \frac{1}{2} \zeta \cdot M_\omega(\theta) \zeta, \theta, L_\omega(\theta) \zeta \right) \quad (4.4)$$

where $\zeta = (z, \bar{z})$, $M_\omega(\theta)$ and $L_\omega(\theta)$ are linear bounded operators from $\ell^{2,p} \times \ell^{2,p}$ into itself for all $p \geq 0$ and $L_\omega(\theta)$ is invertible;

- (iii) a Lipschitz family of new normal forms

$$N^*(\omega) = \sum_{1 \leq j \leq n} \omega_j y_j + \sum_{j \geq 1} \Omega_j^*(\omega) z_j \bar{z}_j;$$

such that on $\Pi_\varepsilon \times \mathcal{P}^0$, $H \circ \Phi = N^*$.

Moreover the new external frequencies are close to the original ones, $\|\Omega^* - \Omega\|_{2\beta, \Pi_\varepsilon} \leq c\varepsilon$, and the new frequencies satisfy a nonresonant condition, i.e.

$$|k \cdot \omega + l \cdot \Omega^*(\omega)| \geq \frac{\alpha}{2} \cdot \frac{\langle l \rangle}{\exp(|k|^{1/\iota})}, \quad \iota \geq 2, (k, l) \in \mathcal{Z},$$

for some $\alpha > 0$ and $\omega \in \Pi_\varepsilon$.

Proof. As [22], Assumption \mathcal{A} is clear. We now check Assumption \mathcal{B} holds. Firstly, we need to check that $(\frac{\partial P}{\partial z_k})_{k \geq 1} \in \ell^{2,2}$ and $(\frac{\partial P_{\omega_j}}{\partial z_k})_{k \geq 1} \in \ell^{2,2}$ with $j = 1, \dots, n$.

Note that $\frac{\partial P}{\partial z_k} = \int_{\mathbb{R}} V(x, \theta; \omega) \bar{u} h_k dx$, which is the k -th coefficient of the decomposition of $V(x, \theta; \omega) \bar{u}$ in the Hermite basis. It follows that $(\frac{\partial P}{\partial z_k})_{k \geq 1} \in \ell^{2,2}$ if and only if $V(x, \theta; \omega) \bar{u} \in \mathcal{H}^2$. From $|V| \leq C$, $|\partial_x V| \leq C$, $|\partial_x^2 V| \leq C$, $\bar{u} \in \mathcal{H}^2$ and a straightforward computation, we have $V(x, \theta; \omega) \bar{u} \in \mathcal{H}^2$. This implies that $(\frac{\partial P}{\partial z_k})_{k \geq 1} \in \ell^{2,2}$. Similarly from $|\partial_{\omega_j} V| \leq C$, $|\partial_x(\partial_{\omega_j} V)| \leq C$, $|\partial_x^2(\partial_{\omega_j} V)| \leq C$ and $\bar{u} \in \mathcal{H}^2$ we obtain $(\frac{\partial P_{\omega_j}}{\partial z_k})_{k \geq 1} \in \ell^{2,2}$ and thus (B1) is satisfied.

In the following we turn to (B2) in Assumption \mathcal{B} . From (1.2), Lemma 1.4 and a straightforward computation,

$$\begin{aligned} \left\| \frac{\partial P}{\partial z_k} \right\|_{D(s,r)} &= \sup_{D(s,r)} \left| \int_{\mathbb{R}} V(x, \theta; \omega) \bar{u} h_k dx \right| \\ &\preceq \left(\int_{\mathbb{R}} \frac{|h_k|^2}{(1 + \ln(1 + x^2))^{2\beta}} dx \right)^{\frac{1}{2}} \cdot \sup_{D(s,r)} \left(\int_{\mathbb{R}} |\bar{u}|^2 dx \right)^{\frac{1}{2}} \preceq \frac{C_\beta r}{(1 + \ln k)^\beta}. \end{aligned}$$

Similarly,

$$\left\| \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l} \right\|_{D(s,r)} = \sup_{D(s,r)} \left| \int_{\mathbb{R}} V(x, \theta; \omega) h_k h_l dx \right| \preceq \frac{C_\beta}{(1 + \ln k)^\beta (1 + \ln l)^\beta}.$$

From the conditions (1.2) - (1.4) and a similar computation we obtain

$$\left\| \frac{\partial P}{\partial z_k} \right\|_{D(s,r)}^{\mathfrak{L}} \preceq \frac{C_\beta r}{(1 + \ln k)^\beta} \quad \text{and} \quad \left\| \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l} \right\|_{D(s,r)}^{\mathfrak{L}} \preceq \frac{C_\beta}{(1 + \ln k)^\beta (1 + \ln l)^\beta}.$$

It follows that $P \in \Gamma_{r,D(s,r)}^\beta$ with $s = \rho$ and $\beta \geq 2(n + 2)$.

For our application to Theorem 1.1 we will choose $M = 2\pi$, $\beta = \iota(n + 2)$ with $\iota \geq 2$. A straight computation shows that

$$\begin{aligned} &\|X_{\varepsilon P}\|_{r,D(\rho,r)} + \langle \varepsilon P \rangle_{r,D(\rho,r)} + \frac{\alpha}{2\pi} (\|X_{\varepsilon P}\|_{r,D(\rho,r)}^{\mathfrak{L}} + \langle \varepsilon P \rangle_{r,D(\rho,r)}^{\mathfrak{L}}) \\ &\preceq \frac{c(\iota, n)\varepsilon}{\rho} (1 + \alpha) \leq \frac{2c(\iota, n)\varepsilon}{\rho} \leq \gamma\alpha^5, \end{aligned}$$

if we choose $\alpha = \varepsilon^{\frac{1}{10}}$ and $\varepsilon \leq \varepsilon_0 := (\frac{\gamma\rho}{2c(\iota,n)})^2$.

Therefore Theorem 2.2 applies with $p = 2$. Following [22] we have:

(i) the symplectic coordinates transformation Φ is quadratic and thus it is defined on the whole phase space and have the form

$$\Phi_\omega(y, \theta, \zeta) = (y + \frac{1}{2}\zeta \cdot M_\omega(\theta)\zeta, \theta, L_\omega(\theta)\zeta);$$

- (ii) the new normal form still have the same frequencies vector ω ;
- (iii) the new Hamiltonian reduces to the new normal form, i.e., $R^* = 0$;
- (iv) the symplectic coordinates transformation Φ_ω , which is defined by Theorem 4.1 on each \mathcal{P}^2 , extends to $\mathcal{P}^0 := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{2,0} \times \ell^{2,0}$.

We complete the proof of Theorem 4.1. Meanwhile, Theorem 1.1 follows directly from Theorem 4.1. \square

Proof of Corollary 1.2. See [22]. \square

Proof of Corollary 1.3. We follow the scheme developed in [22]. Firstly we write the solution $u(t, x)$ of (4.1) with initial datum $u_0(x) = \sum_{j \geq 1} z_j(0) h_j(x)$ as $u(t, x) = \sum_{j \geq 1} z_j(t) h_j(x)$ with

$$(z, \bar{z})(t) = L_\omega(\omega t)(z'(0)e^{-i\Omega^* t}, \bar{z}'(0)e^{i\Omega^* t})$$

and $(z'(0), \bar{z}'(0)) = L_\omega^{-1}(0)(z(0), \bar{z}(0))$. From the structure of $L_\omega(\theta)$, more clearly $(L_\omega(\theta))_{jk}^{12} = (L_\omega(\theta))_{jk}^{21} = 0$, we then have

$$u(t, x) = \sum_{j \geq 1} \psi_j(\omega t, x) e^{-i\Omega_j^* t},$$

where $\psi_j(\theta, x) = \sum_{l \geq 1} (L_\omega(\theta))_{jk}^{11} z'_l(0) h_l(x)$. In particular the solutions are all almost periodic in time with a nonresonant frequencies vector (ω, Ω^*) . By a straight computation we can prove that $\psi_j(\omega t, x) e^{-i\Omega_j^* t}$ solves (4.1) if and only if $k \cdot \omega + \Omega_j^*$ is an eigenvalue of K with eigenfunction $\psi_j(\theta, x) e^{ik \cdot \theta}$. This shows that the spectrum set of the Floquet operator K equals to $\{k \cdot \omega + \Omega_j^* | k \in \mathbb{Z}^n, j \geq 1\}$ and thus we complete the proof. \square

5. PROOF OF KAM THEOREM

5.1. The linearized equation. Let $H = N + P$ be a Hamiltonian, where

$$N = \sum_{1 \leq j \leq n} \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) z_j \bar{z}_j, \quad (5.1)$$

and P satisfies Assumption \mathcal{B} in Sect. 2. The aim in this section is to put $N + P$ into a new normal form $N_+ + P_+$ such that P_+ is much smaller than P . To do this, we need to solve the homological equation

$$\{F, N\} + \hat{N} = R, \quad (5.2)$$

where R is the second order Taylor approximation of P ,

$$R = \sum_{2|m|+|q+\bar{q}| \leq 2} \sum_{k \in \mathbb{Z}^n} R_{kmq\bar{q}} e^{ik\theta} y^m z^q \bar{z}^{\bar{q}} \quad (5.3)$$

with $R_{kmq\bar{q}} = P_{kmq\bar{q}}$, and F has a similar form as R ,

$$F = \sum_{2|m|+|q+\bar{q}| \leq 2} \sum_{k \in \mathbb{Z}^n} F_{kmq\bar{q}} e^{ik\theta} y^m z^q \bar{z}^{\bar{q}}. \quad (5.4)$$

From [38], we have

Lemma 5.1. *Suppose $|\omega|^\mathfrak{L} + \|\Omega\|_{2\beta, \Pi}^\mathfrak{L} \leq M$ uniformly on Π and*

$$|\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| \geq \frac{\langle l \rangle \alpha}{A_k}, \quad (k, l) \in \mathcal{Z},$$

where $\alpha > 0$ and $A_k = e^{|k|^{\tau/\beta}} (\beta > \tau)$. Then the linearized equation $\{F, N\} + \hat{N} = R$ has a solution F , \hat{N} satisfying $[F] = 0$, $\hat{N} = [R] = \sum_{|m|+|q|=1} R_{0mq\bar{q}} y^m z^q \bar{z}^{\bar{q}}$, and

$$\begin{aligned} \|X_{\hat{N}}\|_r &\leq \|X_R\|_r, & \|X_F\|_{r, D(s-\sigma, r)} &\leq \frac{c(n, \beta) e^{3(\frac{4}{\sigma})^{t_1}}}{\alpha} \|X_R\|_r, \\ \|X_{\hat{N}}\|_r^\mathfrak{L} &\leq \|X_R\|_r^\mathfrak{L}, & \|X_F\|_{r, D(s-\sigma, r)}^\mathfrak{L} &\leq \frac{c(n, \beta) e^{3(\frac{4}{\sigma})^{t_1}}}{\alpha} (\|X_R\|_r^\mathfrak{L} + \frac{M}{\alpha} \|X_R\|_r) \end{aligned}$$

for $0 < \sigma \leq s$, $t_1 = \frac{\tau}{\beta - \tau}$ and the short hand $\|\cdot\|_r = \|\cdot\|_{r, D(s, r)}$ is used.

Introduce the space $\Gamma_{r, D(s, r)}^{\beta, +} \subset \Gamma_{r, D(s, r)}^\beta$ endowed with the norm $\langle \cdot \rangle_{r, D(s, r)}^+ + \langle \cdot \rangle_{r, D(s, r)}^{\mathfrak{L}}$ defined by the following conditions:

$$\begin{aligned} \|F\|_{D(s, r)}^* &\leq r^2 \langle F \rangle_{r, D(s, r)}^{+, *}, & \max_{1 \leq j \leq n} \left\| \frac{\partial F}{\partial y_j} \right\|_{D(s, r)}^* &\leq \langle F \rangle_{r, D(s, r)}^{+, *}, \\ \left\| \frac{\partial F}{\partial w_j} \right\|_{D(s, r)}^* &\leq \frac{r}{j(1 + \ln j)^\beta} \langle F \rangle_{r, D(s, r)}^{+, *}, & \forall j \geq 1 \text{ and } w_j = z_j \text{ or } \bar{z}_j, \\ \left\| \frac{\partial^2 F}{\partial w_j \partial w_l} \right\|_{D(s, r)}^* &\leq \frac{\langle F \rangle_{r, D(s, r)}^{+, *}}{(1 + |j - l|)(1 + \ln j)^\beta (1 + \ln l)^\beta}, & \forall j, l \geq 1 \text{ and } w_j = z_j \text{ or } \bar{z}_j. \end{aligned}$$

Lemma 5.2. *Assume that the frequencies satisfy*

$$|\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| \geq \frac{\langle l \rangle \alpha}{A_k}, \quad (k, l) \in \mathcal{Z}, \quad (5.5)$$

where $\alpha > 0$ and $A_k = e^{|k|^{\tau/\beta}}$ ($\beta > \tau$) uniformly on Π . Let F, \hat{N} be given in the above lemma and $R \in \Gamma_{r,D(s,r)}^\beta$, then for any $0 < \sigma < s$, we have $F \in \Gamma_{r,D(s-\sigma,r)}^{\beta,+}$, $\hat{N} \in \Gamma_{r,D(s-\sigma,r)}^\beta$ such that

$$\begin{aligned} \langle F \rangle_{r,D(s-\sigma,r)}^+ &\leq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha} \langle R \rangle_{r,D(s,r)}, \\ \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}} &\leq \frac{c(n,\beta)e^{6(\frac{8}{\sigma})^{t_1}}}{\alpha} \left(\frac{M}{\alpha} \langle R \rangle_{r,D(s,r)} + \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}} \right) \end{aligned}$$

with $t_1 = \frac{\tau}{\beta-\tau}$, and

$$\langle \hat{N} \rangle_{r,D(s-\sigma,r)} \preceq \langle R \rangle_{r,D(s,r)}, \quad \langle \hat{N} \rangle_{r,D(s-\sigma,r)}^{\mathfrak{L}} \preceq \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}}.$$

Proof. Our aim is to solve the homological equation (5.2) and find a solution F for it. A straightforward computation shows that the coefficients in (5.4) are given by

$$iF_{kmq\bar{q}} = \begin{cases} \frac{R_{kmq\bar{q}}}{k \cdot \omega + (q - \bar{q})\Omega}, & \text{if } |k| + |q - \bar{q}| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5.6)$$

As [22], in the following we will use the notation $q_j = (0, \dots, 0, 1, 0, \dots)$, where “1” is in the j -th position and $q_{jl} = q_j + q_l$. The variables z and \bar{z} exactly play the same role, therefore it is enough to study the derivatives in z . We first show that

$$\langle F \rangle_{r,D(s-\sigma,r)}^+ \preceq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha} \langle R \rangle_{r,D(s,r)}.$$

From $|R_{k0q_{jl}0}| \preceq \frac{\langle R \rangle_{r,D(s,r)} e^{-|k|s}}{(1+\ln j)^\beta (1+\ln l)^\beta}$, (5.5) and (5.6),

$$|F_{k0q_{jl}0}| \preceq \frac{A_k \langle R \rangle_{r,D(s,r)} e^{-|k|s}}{\alpha |j+l|(1+\ln j)^\beta (1+\ln l)^\beta} \preceq \frac{A_k \langle R \rangle_{r,D(s,r)} e^{-|k|s}}{\alpha (1+|j-l|)(1+\ln j)^\beta (1+\ln l)^\beta}.$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial^2 F}{\partial z_j \partial \bar{z}_l} \right\|_{D(s-\sigma,r)} &\leq \sum_{k \in \mathbb{Z}^n} |F_{k0q_{jl}0}| e^{|k|(s-\sigma)} \\ &\preceq \sum_{k \in \mathbb{Z}^n} \frac{A_k \langle R \rangle_{r,D(s,r)} e^{-|k|s}}{\alpha (1+|j-l|)(1+\ln j)^\beta (1+\ln l)^\beta} \\ &\preceq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha (1+|j-l|)(1+\ln j)^\beta (1+\ln l)^\beta} \langle R \rangle_{r,D(s,r)}. \end{aligned} \quad (5.7)$$

From Lemma 7.3 and a similar computation, we have

$$\left\| \frac{\partial F}{\partial z_j} \right\|_{D(s-\sigma,r)} \preceq \frac{c(n,\beta)re^{2(\frac{2}{\sigma})^{t_1}}}{\alpha (1+j)(1+\ln j)^\beta} \langle R \rangle_{r,D(s,r)}. \quad (5.8)$$

Similarly,

$$\left\| \frac{\partial F}{\partial y_j} \right\|_{D(s-\sigma,r)} \preceq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha} \langle R \rangle_{r,D(s,r)}, \quad (5.9)$$

and

$$\|F\|_{D(s-\sigma,r)} \preceq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha} r^2 \langle R \rangle_{r,D(s,r)}. \quad (5.10)$$

The above estimates (5.7), (5.8), (5.9) and (5.10) show us that

$$\langle F \rangle_{r,D(s-\sigma,r)}^+ \preceq \frac{c(n,\beta)e^{2(\frac{2}{\sigma})^{t_1}}}{\alpha} \langle R \rangle_{r,D(s,r)}.$$

It remains to check the estimates with the Lipschitz semi-norms. As in [38], for $|k| + |q_j - q_l| \neq 0$ define $\delta_{k,jl} = k \cdot \omega + \Omega_j - \Omega_l$ and $\Delta = \Delta_{\xi\zeta}$. Then we have

$$i\Delta F_{k0q_j\bar{q}_l} = \delta_{k,jl}^{-1}(\eta)\Delta R_{k0q_j\bar{q}_l} + R_{k0q_j\bar{q}_l}(\xi)\Delta\delta_{k,jl}^{-1},$$

and

$$-\Delta\delta_{k,jl}^{-1} = \frac{\langle k, \Delta\omega \rangle + \Delta\Omega_j - \Delta\Omega_l}{\delta_{k,jl}(\xi)\delta_{k,jl}(\zeta)}.$$

By the small divisor assumptions and a direct computation, we have

$$\begin{aligned} \frac{|\Delta F_{k0q_j\bar{q}_l}|}{|\xi - \eta|} &\preceq \frac{A_k}{\alpha\langle j - l \rangle} \frac{|\Delta R_{k0q_j\bar{q}_l}|}{|\xi - \eta|} + \frac{M|k|A_k^2}{\alpha^2\langle j - l \rangle^2} |R_{k0q_j\bar{q}_l}| \\ &\preceq \frac{|k|A_k^2}{\alpha\langle j - l \rangle} \left(\frac{|\Delta R_{k0q_j\bar{q}_l}|}{|\xi - \eta|} + \frac{M}{\alpha} |R_{k0q_j\bar{q}_l}| \right). \end{aligned}$$

Now we go to estimate $\langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}$. We only estimate $\left\| \frac{\partial^2 F}{\partial z_j \partial \bar{z}_l} \right\|_{D(s-\sigma,r)}^{\mathfrak{L}}$. Note $\frac{\partial^2 F}{\partial z_j \partial \bar{z}_l} = \sum_{k \in \mathbb{Z}^n} F_{k0q_j\bar{q}_l} e^{ik\theta}$ and $\Delta \frac{\partial^2 F}{\partial z_j \partial \bar{z}_l} = \sum_{k \in \mathbb{Z}^n} \Delta F_{k0q_j\bar{q}_l} e^{ik\theta}$, it follows that for $(\theta, y, z, \bar{z}) \in D(s - \sigma, r)$,

$$\begin{aligned} \frac{|\Delta \frac{\partial^2 F}{\partial z_j \partial \bar{z}_l}|}{|\xi - \eta|} &\leq \sum_k \frac{|\Delta F_{k0q_j\bar{q}_l}|}{|\xi - \eta|} e^{|k|(s-\sigma)} \\ &\preceq \sum_k \frac{|k|A_k^2}{\alpha\langle j - l \rangle} \left(\frac{|\Delta R_{k0q_j\bar{q}_l}|}{|\xi - \eta|} + \frac{M}{\alpha} |R_{k0q_j\bar{q}_l}| \right) e^{|k|(s-\sigma)} \\ &\preceq \sum_k \frac{|k|A_k^2}{\alpha\langle j - l \rangle} \left(|R_{k0q_j\bar{q}_l}|^{\mathfrak{L}} + \frac{M}{\alpha} |R_{k0q_j\bar{q}_l}| \right) e^{|k|(s-\sigma)}. \end{aligned}$$

Combining with $|R_{k0q_j\bar{q}_l}| \preceq \frac{\langle R \rangle_{r,D(r,s)} e^{-|k|s}}{(1+\ln j)^\beta (1+\ln l)^\beta}$ and $|R_{k0q_j\bar{q}_l}|^{\mathfrak{L}} \preceq \frac{\langle R \rangle_{r,D(r,s)}^{\mathfrak{L}} e^{-|k|s}}{(1+\ln j)^\beta (1+\ln l)^\beta}$, we deduce that

$$\left\| \frac{\partial^2 F}{\partial z_j \partial \bar{z}_l} \right\|_{D(s-\sigma,r)}^{\mathfrak{L}} \preceq \frac{c(n, \beta) e^{6(\frac{s}{\sigma})^{t_1}}}{\alpha(1+\ln j)^\beta (1+\ln l)^\beta (1+|j-l|)} \left(\frac{M}{\alpha} \langle R \rangle_{r,D(s,r)} + \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}} \right).$$

A similar computation for other terms provides that

$$\langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}} \preceq \frac{c(n, \beta) e^{6(\frac{s}{\sigma})^{t_1}}}{\alpha} \left(\frac{M}{\alpha} \langle R \rangle_{r,D(s,r)} + \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}} \right).$$

The estimates for \widehat{N} are similar and we omit the details. \square

Now we turn to the estimates on Poisson bracket.

Lemma 5.3. *Let $R \in \Gamma_{r,D(s,r)}^\beta$ and $F \in \Gamma_{r,D(s-\sigma,r)}^{\beta,+}$ be both of degree 2, i.e., R, F are of the forms (5.3) and (5.4), respectively. Then, for any $0 < 2\sigma < s$,*

$$\langle \{R, F\} \rangle_{r,D(s-2\sigma, \frac{s}{2})} \preceq \frac{1}{\sigma} \langle R \rangle_{r,D(s,r)} \langle F \rangle_{r,D(s-\sigma,r)}^+, \quad (5.11)$$

$$\langle \{R, F\} \rangle_{r,D(s-2\sigma, \frac{s}{2})}^{\mathfrak{L}} \preceq \frac{1}{\sigma} \left(\langle R \rangle_{r,D(s,r)} \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}} + \langle F \rangle_{r,D(s-\sigma,r)}^+ \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}} \right). \quad (5.12)$$

Proof. For simplicity we denote $\langle R \rangle := \langle R \rangle_{r,D(s,r)}$, $\langle R \rangle^{\mathfrak{L}} := \langle R \rangle_{r,D(s,r)}^{\mathfrak{L}}$, $\langle F \rangle^+ := \langle F \rangle_{r,D(s-\sigma,r)}^+$ and $\langle F \rangle^{+,\mathfrak{L}} := \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}$. Note that

$$\{R, F\} = \sum_{k=1}^n \left(\frac{\partial R}{\partial \theta_k} \frac{\partial F}{\partial y_k} - \frac{\partial R}{\partial y_k} \frac{\partial F}{\partial \theta_k} \right) + i \sum_{j \geq 1} \left(\frac{\partial R}{\partial z_j} \frac{\partial F}{\partial \bar{z}_j} - \frac{\partial R}{\partial \bar{z}_j} \frac{\partial F}{\partial z_j} \right),$$

it remains to estimate each term of this expansion and its derivatives.

We first prove (5.11). From Cauchy and the basic inequality $\sum_{j \geq 1} \frac{1}{j(1 + \ln j)^{2\beta}} \preceq 1 (\beta \geq 1)$, we have

$$\|\{R, F\}\|_{D(s-2\sigma, r)} \preceq \frac{r^2 \langle R \rangle \langle F \rangle^+}{\sigma}. \quad (5.13)$$

Similarly,

$$\max_{1 \leq j \leq n} \left\| \frac{\partial}{\partial y_j} \{R, F\} \right\|_{D(s-2\sigma, r)} \preceq \frac{\langle R \rangle \langle F \rangle^+}{\sigma}. \quad (5.14)$$

Write

$$\begin{aligned} \frac{\partial}{\partial z_j} \{R, F\} &= \sum_{k=1}^n \left(\frac{\partial^2 R}{\partial \theta_k \partial z_j} \frac{\partial F}{\partial y_k} + \frac{\partial R}{\partial \theta_k} \frac{\partial^2 F}{\partial y_k \partial z_j} - \frac{\partial^2 R}{\partial y_k \partial z_j} \frac{\partial F}{\partial \theta_k} - \frac{\partial R}{\partial y_k} \frac{\partial^2 F}{\partial \theta_k \partial z_j} \right) \\ &+ i \sum_{k \geq 1} \left(\frac{\partial^2 R}{\partial z_k \partial z_j} \frac{\partial F}{\partial \bar{z}_k} + \frac{\partial R}{\partial z_k} \frac{\partial^2 F}{\partial \bar{z}_k \partial z_j} - \frac{\partial^2 R}{\partial \bar{z}_k \partial z_j} \frac{\partial F}{\partial z_k} - \frac{\partial R}{\partial \bar{z}_k} \frac{\partial^2 F}{\partial z_k \partial z_j} \right) \\ &:= (I) + (II). \end{aligned} \quad (5.15)$$

By a direct computation it holds that

$$\begin{aligned} \|(I)\|_{D(s-2\sigma, \frac{r}{2})} &\leq \sum_{k=1}^n \left(\frac{r \langle R \rangle \langle F \rangle^+}{\sigma(1 + \ln j)^\beta} + \frac{4r \langle R \rangle \langle F \rangle^+}{\sigma j(1 + \ln j)^\beta} + \frac{4r \langle R \rangle \langle F \rangle^+}{\sigma(1 + \ln j)^\beta} + \frac{r \langle R \rangle \langle F \rangle^+}{\sigma j(1 + \ln j)^\beta} \right) \\ &\preceq \frac{r \langle R \rangle \langle F \rangle^+}{\sigma(1 + \ln j)^\beta}. \end{aligned}$$

From Lemma 7.1, $\|(II)\|_{D(s-2\sigma, \frac{r}{2})} \preceq \frac{r \langle R \rangle \langle F \rangle^+}{(1 + \ln j)^\beta}$. Thus

$$\left\| \frac{\partial}{\partial z_j} \{R, F\} \right\|_{D(s-2\sigma, \frac{r}{2})} \preceq \frac{r \langle R \rangle \langle F \rangle^+}{\sigma(1 + \ln j)^\beta}. \quad (5.16)$$

By the same method and Lemma 7.1 we obtain

$$\left\| \frac{\partial^2}{\partial z_j \partial z_l} \{R, F\} \right\|_{D(s-2\sigma, \frac{r}{2})} \preceq \frac{\langle R \rangle \langle F \rangle^+}{\sigma(1 + \ln j)^\beta (1 + \ln l)^\beta}. \quad (5.17)$$

Together with (5.13)-(5.17), (5.11) is proved.

For the Lipschitz norm estimates, we first estimate $\left\| \frac{\partial}{\partial z_j} \{R, F\} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}}$. Note that

$$\left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \leq \left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \left\| \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})} + \left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})} \left\| \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}}$$

where

$$\begin{aligned} \left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})} &\leq \frac{1}{\sigma} \left\| \frac{\partial R}{\partial z_j} \right\|_{D(s, r)} \leq \frac{r \langle R \rangle}{\sigma(1 + \ln j)^\beta}, \\ \left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} &\leq \frac{1}{\sigma} \left\| \frac{\partial R}{\partial z_j} \right\|_{D(s, r)}^{\mathfrak{L}} \leq \frac{r \langle R \rangle^{\mathfrak{L}}}{\sigma(1 + \ln j)^\beta}, \end{aligned}$$

$\left\| \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})} \leq \langle F \rangle^+$ and $\left\| \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \leq \langle F \rangle^{+, \mathfrak{L}}$. Hence

$$\left\| \frac{\partial^2 R}{\partial \theta_k \partial z_j} \frac{\partial F}{\partial y_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \preceq \frac{r(\langle R \rangle^{\mathfrak{L}} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \mathfrak{L}})}{\sigma(1 + \ln j)^\beta}.$$

For the other terms in (I) in (5.15) the estimates are similar. Thus,

$$\|(I)\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \preceq \frac{r(\langle R \rangle^{\mathfrak{L}} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \mathfrak{L}})}{\sigma(1 + \ln j)^\beta}.$$

For (II) we only estimate $\left\| \sum_{k \geq 1} \frac{\partial^2 R}{\partial z_k \partial z_j} \frac{\partial F}{\partial \bar{z}_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}}$. From Lemma 7.1,

$$\begin{aligned} & \left\| \sum_k \frac{\partial^2 R}{\partial z_k \partial z_j} \frac{\partial F}{\partial \bar{z}_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \\ & \leq \sum_k \left\| \frac{\partial^2 R}{\partial z_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \left\| \frac{\partial F}{\partial \bar{z}_k} \right\|_{D(s-2\sigma, \frac{r}{2})} + \sum_k \left\| \frac{\partial^2 R}{\partial z_k \partial z_j} \right\|_{D(s-2\sigma, \frac{r}{2})} \left\| \frac{\partial F}{\partial \bar{z}_k} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \\ & \leq \sum_k \frac{\langle R \rangle^{\mathfrak{L}}}{(1 + \ln k)^\beta (1 + \ln j)^\beta} \frac{r \langle F \rangle^+}{k(1 + \ln k)^\beta} + \sum_k \frac{\langle R \rangle}{(1 + \ln k)^\beta (1 + \ln j)^\beta} \frac{r \langle F \rangle^{+, \mathfrak{L}}}{k(1 + \ln k)^\beta} \\ & \preceq \frac{r}{(1 + \ln j)^\beta} (\langle R \rangle^{\mathfrak{L}} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \mathfrak{L}}). \end{aligned}$$

Similar other estimates result in

$$\|(II)\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \preceq \frac{r}{(1 + \ln j)^\beta} (\langle R \rangle^{\mathfrak{L}} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \mathfrak{L}}).$$

Therefore,

$$\left\| \frac{\partial}{\partial z_j} \{R, F\} \right\|_{D(s-2\sigma, \frac{r}{2})}^{\mathfrak{L}} \preceq \frac{r}{\sigma(1 + \ln j)^\beta} (\langle R \rangle^{\mathfrak{L}} \langle F \rangle^+ + \langle R \rangle \langle F \rangle^{+, \mathfrak{L}}).$$

To obtain (5.12) we need some other estimates and the proofs are similar. We omit them for simplicity. \square

5.2. Phase flow. In this subsection we study the Hamiltonian flow generated by $F \in \Gamma_{r, D(s-\sigma, r)}^{\beta, +}$ which is globally of degree 2. Namely, we consider the system

$$\begin{cases} (\dot{\theta}(t), \dot{y}(t), \dot{z}(t), \dot{\bar{z}}(t)) = X_F(\theta(t), y(t), z(t), \bar{z}(t)), \\ (\theta(0), y(0), z(0), \bar{z}(0)) = (\theta^0, y^0, z^0, \bar{z}^0). \end{cases} \quad (5.18)$$

Lemma 5.4. *Let $0 < 3\sigma < s$ and $F \in \Gamma_{r, D(s-\sigma, r)}^{\beta, +}$ be of degree 2. Assume that*

$$\langle F \rangle_{r, D(s-\sigma, r)}^+ < C\sigma. \quad (5.19)$$

Then the solution of Eq. (5.18) with the initial condition $(\theta^0, y^0, z^0, \bar{z}^0) \in D(s - 3\sigma, \frac{r}{4})$ satisfies $(\theta(t), y(t), z(t), \bar{z}(t)) \in D(s - 2\sigma, \frac{r}{2})$ for all $0 \leq t \leq 1$, and we have the estimates

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial y_k(t)}{\partial w_j^0} \right| \preceq \frac{r}{\sigma(1 + \ln j)^\beta} \langle F \rangle_{r, D(s-\sigma, r)}^+, \quad (5.20)$$

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial w_k(t)}{\partial w_j^0} \right| \preceq \frac{1}{(1 + \ln j)^\beta (1 + \ln k)^\beta (1 + |j - k|)} \langle F \rangle_{r, D(s-\sigma, r)}^+ + \delta_{jk}, \quad (5.21)$$

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial y_k(t)}{\partial y_j^0} \right| \preceq \frac{1}{\sigma} \langle F \rangle_{r, D(s-\sigma, r)}^+ + \delta_{jk}, \quad (5.22)$$

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial^2 y_k(t)}{\partial w_j^0 \partial w_i^0} \right| \preceq \frac{1}{\sigma(1 + \ln j)^\beta (1 + \ln i)^\beta (1 + |j - i|)} \langle F \rangle_{r, D(s-\sigma, r)}^+ \quad (5.23)$$

with $w_k = z_k$ or \bar{z}_k and $w_k^0 = z_k^0$ or \bar{z}_k^0 , $k = 1, 2, \dots$.

Before we give the proof of Lemma 5.4 we introduce a space of infinite dimensional matrices with decaying coefficients. We denote by \mathcal{M} the set of infinite matrices $A : \mathbb{Z}_+ \times \mathbb{Z}_+ \mapsto \mathcal{M}_{2 \times 2}(\mathbb{C})$ with values in the space of complex 2×2 matrices and

$$|A| := \sup_{i,j \geq 1} \|A_{ij}\|_{HS} < \infty,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert Schmidt norm:

$$\|M\|_{HS}^2 := \sum_{k,l=1}^2 |M_{kl}|^2,$$

where $M \in \mathcal{M}_{2 \times 2}(\mathbb{C})$. For $\beta > 0$ we define \mathcal{M}_β the subset of \mathcal{M} such that $[A]_\beta < \infty$, where the norm $[\cdot]_\beta$ is given by the condition

$$\sup_{\xi \in \Pi} \|A_{ij}\|_{HS} \leq \frac{[A]_\beta}{(1 + \ln i)^\beta (1 + \ln j)^\beta (1 + |i - j|)}, \quad i, j \geq 1.$$

In the following lemma we will suppress the parameter ξ for simplicity.

Lemma 5.5. *Let $A, B \in \mathcal{M}_\beta$ where $\xi \in \Pi$. Then $A \cdot B \in \mathcal{M}_\beta$ and $[A \cdot B]_\beta \preceq [A]_\beta [B]_\beta$.*

Proof. For all $j, l \geq 1$, $(A \cdot B)_{jl} = \sum_k A_{jk} B_{kl}$. Thus, for $\xi \in \Pi$,

$$\begin{aligned} \|(A \cdot B)_{jl}\|_{HS} &\leq \sum_{k \geq 1} \|A_{jk}\|_{HS} \|B_{kl}\|_{HS} \\ &\leq \sum_{k \geq 1} \frac{[A]_\beta}{(1 + \ln j)^\beta (1 + \ln k)^\beta (1 + |j - k|)} \frac{[B]_\beta}{(1 + \ln k)^\beta (1 + \ln l)^\beta (1 + |k - l|)} \\ &\leq \frac{[A]_\beta^+ [B]_\beta^+}{(1 + \ln j)^\beta (1 + \ln l)^\beta} \sum_{k \geq 1} \frac{1}{(1 + \ln k)^{2\beta} (1 + |k - l|) (1 + |j - k|)} \\ &\preceq \frac{[A]_\beta [B]_\beta}{(1 + \ln j)^\beta (1 + \ln l)^\beta (1 + |j - l|)}. \end{aligned}$$

The last inequality comes from $\beta \geq 1$, Lemma 7.1 and a similar discussion as Lemma 3.6 in [22]. \square

Proof of Lemma 5.4: For simplicity we introduce the notations $\zeta_j = (z_j, \bar{z}_j)$ and $\zeta = (\zeta_j)_{j \geq 1}$. Then F reads

$$F(\theta, y, \zeta) = a_0(\theta) + a_1(\theta) \cdot y + b(\theta) \cdot \zeta + \frac{1}{2} (B(\theta) \zeta) \cdot \zeta \quad (5.24)$$

with $a_0(\theta) = F(\theta, 0, 0)$, $a_1(\theta) = \nabla_y F(\theta, 0, 0)$, $b(\theta) = \nabla_\zeta F(\theta, 0, 0)$ and $B = (B_{ij})$ is the infinite matrix where

$$B_{ij}(\theta) = \begin{pmatrix} \frac{\partial^2 F}{\partial z_i \partial z_j}(\theta, 0, 0) & \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}(\theta, 0, 0) \\ \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j}(\theta, 0, 0) & \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j}(\theta, 0, 0) \end{pmatrix}. \quad (5.25)$$

The flow X_F^t exists for $0 \leq t \leq 1$ and maps $D(s - 3\sigma, r/4)$ into $D(s - 2\sigma, r/2)$. In the sequel we write

$$(\theta(t), y(t), \zeta(t)) = X_F^t(\theta^0, y^0, \zeta^0). \quad (5.26)$$

From the equation $\dot{\theta} = \nabla_y F(\theta, y, \zeta)$ and (5.19), we have the bound $\sup_{0 \leq t \leq 1} |\Im \theta(t)| < s - 2\sigma$.

We now turn to the equation in ζ . To solve

$$\dot{\zeta}(t) = J \nabla_\zeta F(\theta, y, \zeta)(t) = b_1(t) + \mathcal{B}(t) \zeta(t), \quad \zeta(0) = \zeta_0,$$

where $b_1(t) = Jb(\theta(t))$ and $\mathcal{B}(t) = JB(\theta(t))$ with

$$J = \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}_{j \geq 1},$$

we iterate the integral formulation of the problem

$$\zeta(t) = \zeta^0 + \int_0^t (b_1(t_1) + \mathcal{B}(t_1)\zeta(t_1))dt_1,$$

and formally obtain

$$\zeta(t) = b^\infty(t) + (1 + \mathcal{B}^\infty(t))\zeta^0, \quad (5.27)$$

where

$$b^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} \mathcal{B}(t_j) b_1(t_k) dt_k \cdots dt_1, \quad (5.28)$$

and

$$\mathcal{B}^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k} \mathcal{B}(t_j) dt_k \cdots dt_1. \quad (5.29)$$

It is clear that there exists $C > 0$ so that

$$\sup_{0 \leq t \leq 1} \|\mathcal{B}(t)\|_{\mathcal{L}(\ell^{2,p}, \ell^{2,p})} \leq C,$$

and thus, for all $0 \leq t \leq 1$ the series (5.28) converges by

$$\|b^\infty(t)\|_{\ell^{2,p}} \leq \sup_{0 \leq t \leq 1} \|b_1(t)\|_{\ell^{2,p}} \sum_{k \geq 1} \frac{(4\langle F \rangle^+)^{k-1}}{k!} \leq e \sup_{0 \leq t \leq 1} \|b_1(t)\|_{\ell^{2,p}}.$$

Similarly for $0 \leq t \leq 1$, $\|\mathcal{B}^\infty(t)\|_{\mathcal{L}(\ell^{2,p}, \ell^{2,p})} \leq 4e\langle F \rangle^+$. As a conclusion, the formula (5.27) makes sense. In fact we can say more about $\mathcal{B}^\infty(t)$. For $|\Im \theta| < s - 2\sigma$,

$$|\mathcal{B}_{ij}^{11}| = |\mathcal{B}_{ij}^{21}| = \left| \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j}(\theta, 0, 0) \right| \leq \frac{\langle F \rangle^+}{(1 + \ln i)^\beta (1 + \ln j)^\beta (1 + |i - j|)}.$$

Similar estimates hold for \mathcal{B}_{ij}^{12} , \mathcal{B}_{ij}^{21} and \mathcal{B}_{ij}^{22} . Recall that $\mathcal{B}(t) = JB(\theta(t))$ and $|\Im \theta(t)| < s - 2\sigma$ for $0 \leq t \leq 1$. It follows $\mathcal{B}(t) \in \mathcal{M}_\beta$ and $\sup_{0 \leq t \leq 1} [\mathcal{B}(t)]_\beta \leq \langle F \rangle^+$. Hence, by Lemma 5.5 and (5.29),

$$\sup_{0 \leq t \leq 1} [\mathcal{B}^\infty(t)]_\beta \leq e^{\langle F \rangle^+} - 1 \leq e^{\langle F \rangle_{r,D(s-\sigma,r)}^+}. \quad (5.30)$$

In the following we study the equation in y , $\dot{y}(t) = -\nabla_\theta F(\theta, y, \zeta)(t)$, $y(0) = y_0$. From (5.24),

$$\dot{y}(t) = f(t) + g(t)y(t), \quad y(0) = y_0,$$

where $f(t) = -\nabla_\theta a_0(\theta(t)) - \nabla_\theta b(\theta(t))\zeta - \frac{1}{2}(\nabla_\theta B(\theta(t))\zeta) \cdot \zeta$ and $g(t) = -\nabla_\theta \nabla_y F(\theta, 0, 0)$. As above, we have formally

$$y(t) = f^\infty(t) + (1 + g^\infty(t))y^0, \quad (5.31)$$

where

$$f^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} g(t_j) f(t_k) dt_k \cdots dt_1,$$

and

$$g^\infty(t) = \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k} g(t_j) dt_k \cdots dt_1.$$

From Cauchy we have

$$\sup_{0 \leq t \leq 1} \|g(t)\| \leq \frac{1}{\sigma} \max_{1 \leq j \leq n} \left| \frac{\partial F}{\partial y_j}(\theta(t), 0, 0) \right| \leq \frac{1}{\sigma} \langle F \rangle_{r, D(s-\sigma, r)}^+ := \kappa,$$

which follows that for $0 \leq t \leq 1$,

$$\begin{aligned} \|f^\infty(t)\| &\leq \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} \kappa^{k-1} \|f(t)\| dt_k \cdots dt_1 \\ &\leq \sup_{0 \leq t \leq 1} \|f(t)\| \sum_{k \geq 1} \frac{\kappa^{k-1}}{k!} \leq \sup_{0 \leq t \leq 1} \|f(t)\|. \end{aligned}$$

Similarly for $0 \leq t \leq 1$,

$$\|g^\infty(t)\| \leq \frac{\langle F \rangle^+}{\sigma}. \quad (5.32)$$

Therefore (5.31) makes sense.

Now we turn to show the estimates on the solutions of the equations (5.18). By (5.27),

$$\nabla_{\zeta_j^0} \zeta_k(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_{kj} + \mathcal{B}_{kj}^\infty(t), \quad (5.33)$$

and (5.30) we have (5.21). From $y_k(t) = f_k^\infty(t) + y_k^0 + \sum_{1 \leq j \leq n} g_{jk}^\infty(t) y_j^0$ and (5.32) we obtain (5.22). In the following we give the estimates (5.20) and (5.23).

Since g and g^∞ do not depend on ζ , we obtain that $\frac{\partial y(t)}{\partial z_j^0} = \frac{\partial f^\infty}{\partial z_j^0}$. Now by the definition of f^∞ , we deduce that, for $0 \leq t \leq 1$,

$$\begin{aligned} \left\| \frac{\partial y(t)}{\partial z_j^0} \right\| &= \left\| \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} g(t_j) \frac{\partial f(t_k)}{\partial z_j^0} dt_k \cdots dt_1 \right\| \\ &\leq \sum_{k \geq 1} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{1 \leq j \leq k-1} \|g(t_j)\| \left\| \frac{\partial f(t_k)}{\partial z_j^0} \right\| dt_k \cdots dt_1 \\ &\leq \sum_{k \geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} B^{k-1} \left\| \frac{\partial f(t_k)}{\partial z_j^0} \right\| dt_k \cdots dt_1 \\ &\leq \sup_{0 \leq t \leq 1} |\nabla_{\zeta_j^0} f(t)|. \end{aligned}$$

From a straightforward computation we have, for all $1 \leq l \leq n$,

$$\nabla_{\zeta_k} f_l(t) = -\partial_{\theta_l} b_k(\theta(t)) - \sum_{i \geq 1} \partial_{\theta_l} B_{ki}(\theta(t)) \zeta_i(t), \quad \text{with } b_k(\theta) = \nabla_{\zeta_k} F(\theta, 0, 0). \quad (5.34)$$

By Cauchy we obtain that

$$\sup_{0 \leq t \leq 1} |\partial_{\theta_l} b_k(\theta(t))| \leq \frac{1}{\sigma} \sup_{|\Im \theta| < s-2\sigma} |\nabla_{\zeta_k} F(\theta, 0, 0)| \leq \frac{1}{\sigma} \frac{r \langle F \rangle_{r, D(s-\sigma, r)}^+}{k(1 + \ln k)^\beta}.$$

For the second term in (5.34) we obtain by Cauchy

$$\sup_{0 \leq t \leq 1} \|\partial_{\theta_l} B_{ki}(\theta(t))\| \leq \frac{\langle F \rangle_{r, D(s-\sigma, r)}^+}{\sigma(1 + |k-i|)(1 + \ln k)^\beta(1 + \ln i)^\beta}.$$

Thus,

$$\begin{aligned}
\|\nabla_{\zeta_k} f_l(t)\| &\leq \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma k(1+\ln k)^\beta} + \sum_{i \geq 1} \frac{\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+|k-i|)(1+\ln k)^\beta(1+\ln i)^\beta} \|\zeta_i\| \\
&\leq \frac{\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln k)^\beta} \left(r + \sum_{i \geq 1} \frac{\|\zeta_i\|}{(1+|k-i|)(1+\ln i)^\beta} \right) \\
&\leq \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln k)^\beta}.
\end{aligned}$$

Further, from $\nabla_{\zeta_j^0} f_l(t) = \sum_{k \geq 1} \nabla_{\zeta_j^0} \zeta_k \nabla_{\zeta_k} f_l(t)$, we have

$$\begin{aligned}
\|\nabla_{\zeta_j^0} f_l(t)\| &\leq \sum_{k \geq 1} \|\nabla_{\zeta_j^0} \zeta_k\| \|\nabla_{\zeta_k} f_l(t)\| \\
&\leq \sum_{k \geq 1, k \neq j} \frac{\langle F \rangle_{r,D(s-\sigma,r)}^+}{(1+|k-j|)(1+\ln k)^\beta(1+\ln j)^\beta} \cdot \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln k)^\beta} + \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln j)^\beta} \\
&\leq \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln j)^\beta} \left(1 + \sum_{k \geq 1, k \neq j} \frac{1}{(1+|k-j|)(1+\ln k)^{2\beta}} \right) \\
&\leq \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln j)^\beta}.
\end{aligned}$$

The above third inequality comes from Lemma 7.1 and $\beta \geq 1$. It then follows

$$\sup_{0 \leq t \leq 1} \left\| \frac{\partial y_k(t)}{\partial z_j^0} \right\| \leq \frac{r\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln j)^\beta}.$$

It remains to show (5.23). First we have

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial^2 y_k(t)}{\partial z_j^0 \partial z_i^0} \right| \leq \sup_{0 \leq t \leq 1} \|\nabla_{\zeta_i^0} \nabla_{\zeta_j^0} f(t)\|.$$

Note $\|\nabla_{\zeta_i^0} \nabla_{\zeta_j^0} f(t)\| = \|\nabla_{\theta} B_{ij}(\theta(t))\|$ and use Cauchy in θ ,

$$\left| \frac{\partial^2 y_k(t)}{\partial z_j^0 \partial z_i^0} \right| \leq \frac{\langle F \rangle_{r,D(s-\sigma,r)}^+}{\sigma(1+\ln i)^\beta(1+\ln j)^\beta(1+|i-j|)}.$$

□

Similarly, we have

Lemma 5.6. *Under the assumptions of Lemma 5.4 and the condition $\langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}} \leq C\sigma$, the solution of (5.18) satisfies*

$$\begin{aligned}
\sup_{0 \leq t \leq 1} \left| \frac{\partial y_k(t)}{\partial w_j^0} \right|^{\mathfrak{L}} &\leq \frac{r}{\sigma(1+\ln j)^\beta} \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}, \\
\sup_{0 \leq t \leq 1} \left| \frac{\partial w_k(t)}{\partial w_j^0} \right|^{\mathfrak{L}} &\leq \frac{1}{(1+\ln j)^\beta(1+\ln k)^\beta(1+|j-k|)} \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}, \\
\sup_{0 \leq t \leq 1} \left| \frac{\partial y_k(t)}{\partial y_j^0} \right|^{\mathfrak{L}} &\leq \frac{1}{\sigma} \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}, \\
\sup_{0 \leq t \leq 1} \left| \frac{\partial^2 y_k(t)}{\partial w_j^0 \partial w_i^0} \right|^{\mathfrak{L}} &\leq \frac{1}{\sigma(1+\ln j)^\beta(1+\ln i)^\beta(1+|j-i|)} \langle F \rangle_{r,D(s-\sigma,r)}^{+,\mathfrak{L}}
\end{aligned}$$

with $w_k = z_k$ or \bar{z}_k and $w_k^0 = z_k^0$ or \bar{z}_k^0 , $k = 1, 2, \dots$.

The proof of Lemma 5.4 implies

Corollary 5.7. *The time 1 map X_F^1 reads*

$$\begin{pmatrix} \theta \\ y \\ \zeta \end{pmatrix} \mapsto \begin{pmatrix} K(\theta) \\ L(\theta, \zeta) + M(\theta)\zeta + S(\theta)y \\ T(\theta) + U(\theta)\zeta \end{pmatrix},$$

where $L(\theta, \zeta)$ is quadratic in ζ , $M(\theta)$ and $U(\theta)$ are bounded linear operators from $\ell^{2,p} \times \ell^{2,p}$ into \mathbb{R}^n and $\ell^{2,p} \times \ell^{2,p}$ respectively, and $S(\theta)$ is a bounded linear map from \mathbb{R}^n to \mathbb{R}^n .

5.3. Composition estimates.

Proposition 5.8. *Let $0 < \eta < 1/8$ and $0 < \sigma < s$, $R \in \Gamma_{\eta r, D(s-2\sigma, 4\eta r)}^\beta$ and $F \in \Gamma_{r, D(s-\sigma, r)}^{\beta,+}$ with F of degree 2. Assume that*

$$\langle F \rangle_{r, D(s-\sigma, r)}^+ + \langle F \rangle_{r, D(s-\sigma, r)}^{+, \mathfrak{L}} < C\sigma\eta^2. \quad (5.35)$$

Then $R \circ X_F^1 \in \Gamma_{\eta r, D(s-5\sigma, \eta r)}^\beta$ and

$$\langle R \circ X_F^1 \rangle_{\eta r, D(s-5\sigma, \eta r)} \preceq \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}, \quad (5.36)$$

$$\langle R \circ X_F^1 \rangle_{\eta r, D(s-5\sigma, \eta r)}^{\mathfrak{L}} \preceq \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)} + \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}^{\mathfrak{L}}. \quad (5.37)$$

Remark 5.9. *In Proposition 5.8, we don't require R of degree 2.*

Proof. In the sequel, we use the notation

$$(\theta, y, z, \bar{z}) = X_F^1(\theta^0, y^0, z^0, \bar{z}^0).$$

From (5.35) and $\langle \cdot \rangle_{4\eta r, D(s-\sigma, 4\eta r)} \preceq \eta^{-2} \langle \cdot \rangle_{r, D(s-\sigma, r)}$, it follows

$$\langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)} \preceq \sigma \quad (5.38)$$

which will be used later. Now by (5.35) it is easy to show that X_F^1 maps $D(s-5\sigma, \eta r)$ into $D(s-2\sigma, 4\eta r)$ and thus,

$$\|R \circ X_F^1\|_{D(s-5\sigma, \eta r)} \leq (\eta r)^2 \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}. \quad (5.39)$$

By the Leibniz rule, for all $1 \leq j \leq n$,

$$\frac{\partial(R \circ X_F^1)}{\partial y_j^0} = \sum_{k=1}^n \frac{\partial R(X_F^1)}{\partial y_k} \frac{\partial y_k}{\partial y_j^0}.$$

From the definition $\left\| \frac{\partial R}{\partial y_k} \right\|_{D(s-2\sigma, 4\eta r)} \leq \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}$, and by Lemma 5.4,

$$\sup_{0 \leq t \leq 1} \left| \frac{\partial y_k(t)}{\partial y_j^0} \right| \preceq \frac{1}{\sigma} \langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+ + \delta_{jk}.$$

Thus, from (5.38) we obtain

$$\left| \frac{\partial(R \circ X_F^1)}{\partial y_j^0} \right| \preceq \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}. \quad (5.40)$$

For $j \geq 1$, the derivatives in z_j^0 reads

$$\frac{\partial(R \circ X_F^1)}{\partial z_j^0} = \sum_{k=1}^n \frac{\partial R(X_F^1)}{\partial y_k} \frac{\partial y_k}{\partial z_j^0} + \sum_{k \geq 1} \left(\frac{\partial R(X_F^1)}{\partial z_k} \frac{\partial z_k}{\partial z_j^0} + \frac{\partial R(X_F^1)}{\partial \bar{z}_k} \frac{\partial \bar{z}_k}{\partial z_j^0} \right) := (I) + (II).$$

From (5.38),

$$\begin{aligned} |(I)| &\preceq \sum_{k=1}^n \frac{\eta r}{\sigma(1+\ln j)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)} \langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+ \\ &\preceq \frac{\eta r}{(1+\ln j)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}, \end{aligned}$$

and

$$\begin{aligned} |(II)| &\leq \sum_{k \geq 1} \left(\left\| \frac{\partial R(X_F^1)}{\partial z_k} \right\| \left\| \frac{\partial z_k}{\partial z_j^0} \right\| + \left\| \frac{\partial R(X_F^1)}{\partial \bar{z}_k} \right\| \left\| \frac{\partial \bar{z}_k}{\partial z_j^0} \right\| \right) \\ &\preceq \sum_{k \geq 1} \left(\frac{\langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+}{(1+\ln j)^\beta (1+\ln k)^\beta (1+|j-k|)} + \delta_{jk} \right) \frac{\eta r}{(1+\ln k)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)} \\ &\preceq \frac{\eta r}{(1+\ln j)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)} \left(1 + \langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+ \sum_{k \geq 1} \frac{1}{(1+\ln k)^{2\beta} (1+|j-k|)} \right) \\ (5.38) \quad &\preceq \frac{\eta r}{(1+\ln j)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}. \end{aligned}$$

Thus

$$\left| \frac{\partial(R \circ X_F^1)}{\partial z_j^0} \right| \preceq \frac{\eta r}{(1+\ln j)^\beta} \langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}. \quad (5.41)$$

We now estimate $\left\| \frac{\partial^2(R \circ X_F^1)}{\partial z_i^0 \partial z_j^0} \right\|_{D(s-5\sigma, \eta r)}$. The derivatives reads

$$\frac{\partial^2(R \circ X_F^1)}{\partial z_i^0 \partial z_j^0} = (I_1) + (I_2) + (I_3) + (I_4)$$

with

$$\begin{aligned} (I_1) &= \sum_{k, l=1}^n \frac{\partial^2 R(X_F^1)}{\partial y_k \partial y_l} \frac{\partial y_l}{\partial z_i^0} \frac{\partial y_k}{\partial z_j^0}, \quad (I_2) = \sum_{k=1}^n \frac{\partial R(X_F^1)}{\partial y_k} \frac{\partial^2 y_k}{\partial z_i^0 \partial z_j^0}, \\ (I_3) &= \sum_{k \geq 1} \sum_{l=1}^n \frac{\partial^2 R(X_F^1)}{\partial y_l \partial z_k} \frac{\partial y_l}{\partial z_i^0} \frac{\partial z_k}{\partial z_j^0} + \sum_{k \geq 1} \sum_{p \geq 1} \frac{\partial^2 R(X_F^1)}{\partial z_p \partial z_k} \frac{\partial z_p}{\partial z_i^0} \frac{\partial z_k}{\partial z_j^0} + \sum_{k \geq 1} \sum_{p \geq 1} \frac{\partial^2 R(X_F^1)}{\partial \bar{z}_p \partial z_k} \frac{\partial \bar{z}_p}{\partial z_i^0} \frac{\partial z_k}{\partial z_j^0}, \\ &= (I)_a + (I)_b + (I)_c. \end{aligned}$$

and

$$(I_4) = \sum_{k \geq 1} \left(\sum_{l=1}^n \frac{\partial^2 R(X_F^1)}{\partial y_l \partial \bar{z}_k} \frac{\partial y_l}{\partial z_i^0} + \sum_{p \geq 1} \frac{\partial^2 R(X_F^1)}{\partial z_p \partial \bar{z}_k} \frac{\partial z_p}{\partial z_i^0} + \sum_{p \geq 1} \frac{\partial^2 R(X_F^1)}{\partial \bar{z}_p \partial \bar{z}_k} \frac{\partial \bar{z}_p}{\partial z_i^0} \right) \frac{\partial \bar{z}_k}{\partial z_j^0}.$$

We give a detailed estimation for (I_3) . From Cauchy, (5.38) and Lemma 5.4,

$$\begin{aligned} \|(I)_a\|_{D(s-5\sigma, \eta r)} &\leq \sum_{k \geq 1} \sum_{l=1}^n \left| \frac{\partial^2 R \circ X_F^1}{\partial y_l \partial z_k} \right| \cdot \left| \frac{\partial y_l}{\partial z_i^0} \right| \cdot \left| \frac{\partial z_k}{\partial z_j^0} \right| \\ &\preceq \sum_{k \geq 1} \sum_{l=1}^n (\eta r)^{-2} \left| \frac{\partial R}{\partial z_k} \right|_{D(s-2\sigma, 4\eta r)} \cdot \frac{\eta r \langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+}{\sigma(1+\ln i)^\beta} \cdot \left(\frac{\langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+}{(1+\ln j)^\beta (1+\ln k)^\beta (1+|j-k|)} + \delta_{jk} \right) \\ &\preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1+\ln i)^\beta (1+\ln j)^\beta}. \end{aligned}$$

By the same way,

$$\begin{aligned}
\|(I)_b\|_{D(s-5\sigma, \eta r)} &\preceq \sum_{k,p \geq 1} \left| \frac{\partial^2 R \circ X_F^1}{\partial z_p \partial z_k} \right| \cdot \left| \frac{\partial z_p}{\partial z_i^0} \right| \cdot \left| \frac{\partial z_k}{\partial z_j^0} \right| \\
&\preceq \sum_{k,p \geq 1} \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln k)^\beta (1 + \ln p)^\beta} \cdot \left(\frac{\langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+}{(1 + \ln p)^\beta (1 + \ln i)^\beta (1 + |p - i|)} + \delta_{pi} \right) \\
&\quad \cdot \left(\frac{\langle F \rangle_{4\eta r, D(s-\sigma, 4\eta r)}^+}{(1 + \ln j)^\beta (1 + \ln k)^\beta (1 + |j - k|)} + \delta_{jk} \right) \\
&\preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}.
\end{aligned}$$

Similarly, we have

$$\|(I)_c\|_{D(s-5\sigma, \eta r)} \preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}.$$

Therefore,

$$\|(I_3)\|_{D(s-5\sigma, \eta r)} \preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}.$$

The similar computation provides us

$$\begin{aligned}
\|(I_1)\|_{D(s-5\sigma, \eta r)} &\preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}, \\
\|(I_2)\|_{D(s-5\sigma, \eta r)} &\preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}, \\
\|(I_4)\|_{D(s-5\sigma, \eta r)} &\preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}.
\end{aligned}$$

It results in

$$\left\| \frac{\partial^2 (R \circ X_F^1)}{\partial z_i^0 \partial z_j^0} \right\|_{D(s-5\sigma, \eta r)} \preceq \frac{\langle R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}}{(1 + \ln i)^\beta (1 + \ln j)^\beta}. \quad (5.42)$$

By (5.39), (5.40), (5.41) and (5.42), (5.36) holds. We omit the proof of (5.37), which is similar by using the estimates of Lemma 5.6 instead. \square

Lemma 5.10. *Assume P satisfies Assumption \mathcal{B} and consider its Taylor approximation R of the form (5.3). Then, for all $\eta > 0$,*

$$\|X_R\|_{r, D(s, r)}^* \preceq \|X_P\|_{r, D(s, r)}^*,$$

and

$$\|X_P - X_R\|_{\eta r, D(s, 4\eta r)}^* \preceq \eta \|X_P\|_{r, D(s, r)}^*.$$

We have an analogous result for the norm $\langle \cdot \rangle_{r, D(s, r)}$.

Lemma 5.11. *Let $P \in \Gamma_{r, D(s, r)}^\beta$ and consider its Taylor approximation R of the form (5.3). Then, for all $\eta > 0$,*

$$\langle R \rangle_{\eta r, D(s, r)}^* \preceq \langle P \rangle_{r, D(s, r)}^*,$$

and

$$\langle P - R \rangle_{\eta r, D(s, 4\eta r)}^* \preceq \eta \langle P \rangle_{r, D(s, r)}^*.$$

We omit the proofs for the above two lemmas.

5.4. The KAM Step. Let N be a Hamiltonian in normal form as in (5.1), which reads in the variables (θ, y, z, \bar{z}) ,

$$N = \sum_{1 \leq j \leq n} \omega_j(\xi) + \sum_{j \geq 1} \Omega_j(\xi) z_j \bar{z}_j,$$

and suppose that Assumption \mathcal{A} is satisfied. Consider a perturbation P which satisfies Assumption \mathcal{B} for some $r, s > 0$. Then choose $0 < \eta < 1/8$, $0 < \sigma < s$ and assume that

$$\langle P \rangle_{r,D(s,r)} + \|X_P\|_{r,D(s,r)} + \frac{\alpha}{M} \left(\langle P \rangle_{r,D(s,r)}^{\mathfrak{L}} + \|X_P\|_{r,D(s,r)}^{\mathfrak{L}} \right) \leq \frac{\alpha^2 \eta^2 e^{-7(\frac{s}{\sigma})^{t_1}}}{M c_0}, \quad (5.43)$$

where $t_1 = \frac{\tau}{\beta - \tau}$, c_0 is a large constant depending only on n, τ and β .

5.4.1. Estimates on the new error term. We estimate the new error term P_+ given by the formula

$$P_+ = (P - R) \circ X_F^1 + \int_0^1 \{R(t), F\} \circ X_F^t dt, \quad (5.44)$$

where $R(t) = (1 - t)\hat{N} + tR$.

Lemma 5.12. *Assume (5.43). Then there exists $c(n, \beta) > 0$ so that for all $0 \leq \lambda \leq \alpha/M$,*

$$\begin{aligned} & \langle P_+ \rangle_{\eta r, D(s-5\sigma, \eta r)}^\lambda + \|X_{P_+}\|_{\eta r, D(s-5\sigma, \eta r)}^\lambda \\ & \preceq \frac{c(n, \beta) e^{7(\frac{s}{\sigma})^{t_1}}}{\alpha \eta^2} \left(\langle P \rangle_{r, D(s, r)}^\lambda + \|X_P\|_{r, D(s, r)}^\lambda \right)^2 + \eta \left(\langle P \rangle_{r, D(s, r)}^\lambda + \|X_P\|_{r, D(s, r)}^\lambda \right). \end{aligned}$$

We divide it into two lemmas. From [38], we have

Lemma 5.13. *Assume (5.43), then*

$$\|X_{P_+}\|_{\eta r, D(s-5\sigma, \eta r)}^\lambda \preceq \frac{c(n, \beta) e^{7(\frac{s}{\sigma})^{t_1}}}{\alpha \eta^2} (\|X_P\|_{r, D(s, r)}^\lambda)^2 + \eta \|X_P\|_{r, D(s, r)}^\lambda.$$

Lemma 5.14. *Assume (5.43), then*

$$\langle P_+ \rangle_{\eta r, D(s-5\sigma, \eta r)}^\lambda \preceq \frac{c(n, \beta) e^{7(\frac{s}{\sigma})^{t_1}}}{\alpha \eta^2} (\langle P \rangle_{r, D(s, r)}^\lambda)^2 + \eta \langle P \rangle_{r, D(s, r)}^\lambda.$$

Proof. By (5.44), Proposition 5.8 and Lemma 5.11, we have

$$\begin{aligned} \langle (P - R) \circ X_F^1 \rangle_{\eta r, D(s-5\sigma, \eta r)}^\lambda &= \langle (P - R) \circ X_F^1 \rangle_{\eta r, D(s-5\sigma, \eta r)}^\lambda + \lambda \langle (P - R) \circ X_F^1 \rangle_{\eta r, D(s-5\sigma, \eta r)}^{\mathfrak{L}} \\ &\preceq \langle P - R \rangle_{\eta r, D(s-2\sigma, 4\eta r)}^\lambda \\ &\preceq \eta \langle P \rangle_{r, D(s, r)}^\lambda. \end{aligned}$$

On the other hand, by the same method,

$$\left\langle \int_0^1 \{R(t), F\} \circ X_F^t dt \right\rangle_{\eta r, D(s-5\sigma, \eta r)}^\lambda \preceq \langle \{R(t), F\} \rangle_{\eta r, D(s-2\sigma, 4\eta r)}^\lambda. \quad (5.45)$$

Note $R(t) = (1 - t)\hat{N} + tR$ and $\hat{N} = [R]$, from Lemma 5.3 we obtain

$$\begin{aligned} \langle \{[R], F\} \rangle_{\eta r, D(s-2\sigma, 4\eta r)}^\lambda &\leq \eta^{-2} \langle \{[R], F\} \rangle_{r, D(s-2\sigma, \frac{s}{\sigma})}^\lambda \\ &\preceq \frac{1}{\sigma \eta^2} \left(\langle [R] \rangle_{r, D(s, r)}^\lambda \langle F \rangle_{r, D(s-\sigma, r)}^+ + \langle [R] \rangle_{r, D(s, r)} \langle F \rangle_{r, D(s-\sigma, r)}^{+, \lambda} \right), \end{aligned}$$

where we use $\langle \cdot \rangle_{\eta r, D(s-2\sigma, 4\eta r)} \leq \eta^{-2} \langle \cdot \rangle_{r, D(s-2\sigma, r)}$ for $0 < \eta < 1$. Similarly

$$\langle \{R, F\} \rangle_{\eta r, D(s-2\sigma, 4\eta r)}^\lambda \preceq \frac{1}{\sigma \eta^2} \left(\langle R \rangle_{r, D(s, r)}^\lambda \langle F \rangle_{r, D(s-\sigma, r)}^+ + \langle R \rangle_{r, D(s, r)} \langle F \rangle_{r, D(s-\sigma, r)}^{+, \lambda} \right).$$

Thus, by Lemma 5.2 and $0 \leq \lambda \leq \alpha/M$,

$$\begin{aligned}
(5.45) \quad & \leq \frac{1}{\sigma\eta^2} \left(\langle P \rangle_{r,D(s,r)}^\lambda \langle F \rangle_{r,D(s-\sigma,r)}^+ + \langle P \rangle_{r,D(s,r)} \langle F \rangle_{r,D(s-\sigma,r)}^{+, \lambda} \right) \\
& \leq \frac{c(n, \beta)}{\alpha\sigma\eta^2} \langle P \rangle_{r,D(s,r)}^\lambda \langle P \rangle_{r,D(s,r)} e^{2(\frac{2}{\sigma})^{t_1}} + \lambda \frac{c(n, \beta)M}{\alpha^2\sigma\eta^2} \langle P \rangle_{r,D(s,r)}^2 e^{6(\frac{8}{\sigma})^{t_1}} + \lambda \frac{c(n, \beta)}{\alpha\sigma\eta^2} \langle P \rangle_{r,D(s,r)} \langle P \rangle_{r,D(s,r)}^\mathfrak{L} e^{6(\frac{8}{\sigma})^{t_1}} \\
& \leq \frac{c(n, \beta)}{\alpha\eta^2} e^{7(\frac{8}{\sigma})^{t_1}} (\langle P \rangle_{r,D(s,r)}^\lambda)^2.
\end{aligned}$$

□

5.4.2. Estimates on the frequencies.

Lemma 5.15. *There exists K and $0 < \alpha_+ < \alpha$ so that*

$$|\langle k, \omega_+(\xi) \rangle + \langle l, \Omega_+(\xi) \rangle| \geq \frac{\langle l \rangle \alpha_+}{A_k}, \quad |k| \leq K, \quad |l| \leq 2,$$

where $A_k = e^{|k|^{\tau/\beta}}$ ($\beta > \tau$).

Proof. Note that $\omega_+ = \omega + \widehat{\omega}$, $\Omega_+ = \Omega + \widehat{\Omega}$. Since $\widehat{\omega}_j(\xi) = \frac{\partial \widehat{N}}{\partial y_j}(0, 0, 0, 0, \xi)$, we obtain that

$$|\widehat{\omega}|_\Pi \leq \sup_{D(s,r) \times \Pi} \left| \frac{\partial \widehat{N}}{\partial y} \right| \leq \|X_{\widehat{N}}\|_{r,D(s,r)} \preceq \|X_P\|_{r,D(s,r)}.$$

On the other hand, $\widehat{\Omega}_j(\xi) = \frac{\partial^2 \widehat{N}}{\partial z_j \partial \bar{z}_j}(0, 0, 0, 0, \xi)$, thus

$$\|\widehat{\Omega}\|_{2\beta, \Pi} \leq \sup_{D(s-\sigma, r) \times \Pi} \left| \frac{\partial^2 \widehat{N}}{\partial z_j \partial \bar{z}_j} \right| (1 + \ln j)^{2\beta} \leq \langle \widehat{N} \rangle_{r,D(s-\sigma, r)} \preceq \langle P \rangle_{r,D(s, r)}.$$

Therefore,

$$|\widehat{\omega}|_\Pi + \|\widehat{\Omega}\|_{2\beta, \Pi} \preceq \|X_P\|_{r,D(s, r)} + \langle P \rangle_{r,D(s, r)}.$$

Similarly, for the Lipschitz norms we obtain

$$|\widehat{\omega}|_\Pi^\mathfrak{L} + \|\widehat{\Omega}\|_{2\beta, \Pi}^\mathfrak{L} \preceq \|X_P\|_{r,D(s, r)}^\mathfrak{L} + \langle P \rangle_{r,D(s, r)}^\mathfrak{L}. \quad (5.46)$$

Discussing different cases we easily obtain

$$|\langle k, \widehat{\omega} \rangle + \langle l, \widehat{\Omega} \rangle| \preceq |k| (\|X_P\|_{r,D(s, r)} + \langle P \rangle_{r,D(s, r)}).$$

If we choose $\widehat{\alpha} \geq CK \max_{|k| \leq K} A_K (\|X_P\|_{r,D(s, r)} + \langle P \rangle_{r,D(s, r)})$, then for $|k| \leq K$,

$$\begin{aligned}
|\langle k, \omega_+(\xi) \rangle + \langle l, \Omega_+(\xi) \rangle| & \geq |\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| - |\langle k, \widehat{\omega}(\xi) \rangle + \langle l, \widehat{\Omega}(\xi) \rangle| \\
& \geq \frac{\langle l \rangle \alpha}{A_k} - C|k| (\|X_P\|_{r,D(s, r)} + \langle P \rangle_{r,D(s, r)}) \\
& \geq \frac{\langle l \rangle \alpha_+}{A_k}
\end{aligned}$$

with $\alpha_+ = \alpha - \widehat{\alpha}$.

It remains to show that $\alpha_+ > 0$. This will be done in the KAM iteration below (see (5.49)). □

5.4.3. *The iterative lemma.* Denote $P_0 = P$ and $N_0 = N$. Then at the ν -th step of the Newton scheme, we have a Hamiltonian $H_\nu = N_\nu + P_\nu$ where

$$N_\nu = \sum_{j=1}^n \omega_{\nu,j}(\xi) y_j + \sum_{j \geq 1} \Omega_{\nu,j}(\xi) z_j \bar{z}_j.$$

We will show that there exists a symplectic coordinates transformation $\Phi_{\nu+1} : D_{\nu+1} \times \Pi_{\nu+1} \mapsto D_\nu$ such that $H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1}$ satisfies the same assumptions with $\nu + 1$ in place of ν , where the new normal form $N_{\nu+1}$ is associated with the new frequencies given by $\omega_{\nu+1,j} = \omega_{\nu,j} + \hat{\omega}_{\nu,j}$, $\Omega_{\nu+1,j} = \Omega_{\nu,j} + \hat{\Omega}_{\nu,j}$ and $P_{\nu+1}$ is given by

$$P_{\nu+1} = (P_\nu - R_\nu) \circ X_{F_\nu}^1 + \int_0^1 \{R_\nu(t), F_\nu\} \circ X_{F_\nu}^t dt$$

with $R_\nu(t) = (1-t)\hat{N}_\nu + tR_\nu$.

Let c_1 be twice the maximum of all constants obtained during the KAM step. Set $r_0 = r$, $s_0 = s$, $\alpha_0 = \alpha$ and $M_0 = M$. For $\nu \geq 0$ set

$$\alpha_\nu = \frac{\alpha_0}{2}(1 + 2^{-\nu}), \quad M_\nu = M_0(2 - 2^{-\nu}), \quad \lambda_\nu = \frac{\alpha_\nu}{M_\nu},$$

and

$$\varepsilon_{\nu+1} = \frac{c_1 \varepsilon_\nu^{\frac{133}{100}}}{\alpha_\nu^{\frac{1}{3}}}, \quad \sigma_\nu = \frac{8 \cdot 700^{\iota-1}}{|\ln \varepsilon_\nu|^{\iota-1}}, \quad \eta_\nu^3 = \frac{\varepsilon_\nu^{\frac{99}{100}}}{\alpha_\nu}, \quad s_{\nu+1} = s_\nu - 5\sigma_\nu, \quad r_{\nu+1} = \eta_\nu r_\nu, \quad \beta = \iota\tau (\iota \geq 2),$$

and $D_\nu = D(s_\nu, r_\nu)$.

The initial conditions are chosen in the following way: $\sigma_0 = s_0/48 \leq 1$ so that $s_0 > s_1 > \dots \geq s_0/2$, and assume $\varepsilon_0 \leq \gamma_0 \alpha_0^5$ with $\gamma_0 \leq \min\{(\frac{1}{8Mc_0})^4, c_2(s_0), (\frac{1}{4c_1})^{10}\}$ where $c_2(s_0) = \exp\{-\frac{48 \cdot 8 \cdot 700^{\iota-1}}{s_0}\}$. Furthermore, we define $K_\nu = K_0(\frac{36}{25})^\nu$ with $K_0 = \frac{1}{4} \ln^2(\frac{1}{4c_1\gamma_0})$.

Lemma 5.16. (*Iterative lemma*). Suppose that $H_\nu = N_\nu + P_\nu$ is given on $D_\nu \times \Pi_\nu$, where $N_\nu = \sum_{1 \leq j \leq n} \omega_{\nu,j}(\xi) y_j + \sum_{j \geq 1} \Omega_{\nu,j}(\xi) z_j \bar{z}_j$ is a normal form satisfying

$$|\omega_\nu|_{\Pi_\nu}^\xi + \|\Omega_\nu\|_{2\beta, \Pi_\nu}^\xi \leq M_\nu, \quad (5.47)$$

$$|\langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle| \geq \frac{\langle l \rangle \alpha_\nu}{A_k}, \quad (k, l) \in \mathcal{Z},$$

on Π_ν and

$$\langle P \rangle_{r_\nu, D_\nu}^{\lambda_\nu} + \|X_P\|_{r_\nu, D_\nu}^{\lambda_\nu} \leq \varepsilon_\nu. \quad (5.48)$$

Then there exists a Lipschitz family of real analytic symplectic coordinates transformations $\Phi_{\nu+1} : D_{\nu+1} \times \Pi_{\nu+1} \mapsto D_\nu$ with a closed subset $\Pi_{\nu+1} = \Pi_\nu \setminus \bigcup_{|k| > K_\nu} \mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1})$ of Π_ν , where

$$\mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1}) = \left\{ \xi \in \Pi_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle + \langle l, \Omega_{\nu+1}(\xi) \rangle| < \frac{\langle l \rangle \alpha_{\nu+1}}{A_k} \right\}$$

such that for $H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1}$, the same assumptions (5.47) and (5.48) are satisfied with $\nu + 1$ in place of ν .

Proof. By induction one verifies that

$$\varepsilon_\nu \leq \frac{\alpha_\nu^2 \eta_\nu^2 e^{-7(\frac{8}{\sigma_\nu})^{t_1}}}{M_\nu c_0}, \quad t_1 = \frac{\tau}{\beta - \tau} = \frac{1}{\iota - 1}.$$

So the smallness condition (5.43) at the ν -th KAM step is satisfied, and there exists a transformation $\Phi_{\nu+1} : D_{\nu+1} \times \Pi_{\nu+1} \mapsto D_\nu$ taking H_ν into $H_{\nu+1} = N_{\nu+1} + P_{\nu+1}$. From Lemma 5.12, the new error term satisfies the estimate

$$\langle P_{\nu+1} \rangle_{r_{\nu+1}, D_{\nu+1}}^{\lambda_{\nu+1}} + \|X_{P_{\nu+1}}\|_{r_{\nu+1}, D_{\nu+1}}^{\lambda_{\nu+1}} \leq \frac{c_1}{2} \left(\frac{e^{7 \cdot (\frac{8}{\sigma_\nu})^{t_1}}}{\alpha_\nu \eta_\nu^2} \varepsilon_\nu^2 + \eta_\nu \varepsilon_\nu \right) \leq \varepsilon_{\nu+1}.$$

In view of (5.47) the Lipschitz semi-norm of the new frequencies is bounded by

$$\begin{aligned} |\omega_{\nu+1}|_{\Pi_\nu}^\Sigma + \|\Omega_{\nu+1}\|_{2\beta, \Pi_\nu}^\Sigma &\leq M_\nu + |\widehat{\omega}_\nu|_{\Pi_\nu}^\Sigma + \|\widehat{\Omega}_\nu\|_{2\beta, \Pi_\nu}^\Sigma \\ &\leq M_\nu + \frac{c_1}{2} \varepsilon_\nu \frac{M_\nu}{\alpha_\nu} \leq M_\nu (1 + \frac{1}{2^{\nu+2}}) \\ &\leq M_{\nu+1}, \end{aligned}$$

where the second inequality is from (5.46).

Finally, one verifies that $\alpha_\nu - \alpha_{\nu+1} \geq c_1 K_\nu e^{K_\nu^{\tau/\beta}} \varepsilon_\nu$, hence

$$\alpha_\nu - \alpha_{\nu+1} \geq c_1 K_\nu e^{K_\nu^{\tau/\beta}} (\langle P \rangle_{r_\nu, D_\nu}^{\lambda_\nu} + \|X_P\|_{r_\nu, D_\nu}^{\lambda_\nu}). \quad (5.49)$$

Therefore, by Lemma 5.15, the small divisor estimates hold for the new frequencies with parameter $\alpha_{\nu+1}$ up to $|k| \leq K_\nu$. Removing from Π_ν the union of the resonance zones $\mathcal{R}_{kl}^{\nu+1}(\alpha_{\nu+1})$ for $|k| > K_\nu$ we obtain the parameter domain $\Pi_{\nu+1} \subset \Pi_\nu$ with the required properties. \square

5.4.4. Proof of Theorem 2.2. We follow the proofs in [22] and [38]. For readers' convenience, we use the same notations as in [22]. Firstly as [38], we have the estimates.

Lemma 5.17. *For $\nu \geq 0$,*

$$\begin{aligned} \frac{1}{\sigma_\nu} \|\Phi_{\nu+1} - id\|_{r_\nu, D_{\nu+1}}^{\lambda_\nu}, \quad \|D\Phi_{\nu+1} - I\|_{r_\nu, r_\nu, D_{\nu+1}}^{\lambda_\nu} &\leq c_1 e^{4(\frac{4}{\sigma_\nu})^{t_1}} \alpha_\nu^{-1} \varepsilon_\nu, \\ |\omega_{\nu+1} - \omega_\nu|_{\Pi_\nu}^{\lambda_\nu}, \quad \|\Omega_{\nu+1} - \Omega_\nu\|_{2\beta, \Pi_\nu}^{\lambda_\nu} &\leq c_1 \varepsilon_\nu. \end{aligned}$$

Now suppose the assumptions of Theorem 2.2 are satisfied. To apply the iterative lemma (Lemma 5.16) with $\nu = 0$, set $s_0 = s$, $r_0 = r$, ..., $N_0 = N$, $P_0 = P$ and $\gamma_0 = \gamma$, $\alpha_0 = \alpha$, $M_0 = M$. The smallness condition is satisfied, because

$$\varepsilon = \langle P \rangle_{r_0, D_0}^{\lambda_0} + \|X_P\|_{r_0, D_0}^{\lambda_0} \leq \gamma_0 \alpha_0^5 = \varepsilon_0. \quad (5.50)$$

The small divisor conditions are satisfied by setting $\Pi_0 = \Pi \setminus \bigcup_{k,l} \mathcal{R}_{kl}^0(\alpha_0)$. Then the iterative lemma applies, and we obtain a decreasing sequence of domains $D_\nu \times \Pi_\nu$ and transformations $\Phi^\nu = \Phi_1 \circ \dots \circ \Phi_\nu : D_\nu \times \Pi_{\nu-1} \rightarrow D_{\nu-1}$ for $\nu \geq 1$, such that $H \circ \Phi^\nu = N_\nu + P_\nu$. Moreover the estimates in Lemma 5.17 hold.

From Lemma 5.17 we have $\|D\Phi_\nu\|_{r_{\nu-1}, r_{\nu-1}, D_\nu}^{\lambda_\nu} \leq 1 + c_1 e^{4(\frac{4}{\sigma_{\nu-1}})^{t_1}} \alpha_{\nu-1}^{-1} \varepsilon_{\nu-1}$, and thus

$$\|D\Phi^\nu\|_{r_0, r_\nu, D_\nu} \leq \Pi_{n=0}^\infty (1 + 2^{-n-2}) \leq 2, \quad (5.51)$$

for all $\nu \geq 0$. Similarly we have $\|D\Phi^\nu\|_{r_0, r_\nu, D_\nu}^\Sigma \leq 2$. Thus, $\|\Phi^{\nu+1} - \Phi^\nu\|_{r_0, D_{\nu+1}}^{\lambda_0} \leq \|\Phi_{\nu+1} - id\|_{r_\nu, D_{\nu+1}}^{\lambda_\nu}$. So Φ^ν converge uniformly on $\bigcap D_\nu \times \Pi_\nu = D(s/2) \times \Pi_\alpha$ to a Lipschitz continuous family of real analytic torus embeddings $\Phi : \mathbb{T}^n \times \Pi_\alpha \rightarrow \mathcal{P}^p$.

From (5.51) we obtain $\|\Phi^{\nu+1} - \Phi^\nu\|_{r_0, D_{\nu+1}} \leq 2c_1 \alpha_\nu^{-1} \varepsilon_\nu^{\frac{99}{100}}$. It follows

$$\|\Phi^{\nu+1} - id\|_{r_0, D(s/2) \times \Pi_\alpha} \leq \sum_{n=0}^\nu 2c_1 \alpha_n^{-1} \varepsilon_n^{\frac{99}{100}} \leq \alpha_0^{-1} \varepsilon_0^{\frac{99}{100}} \leq \varepsilon_0^{\frac{1}{2}}.$$

Notice (5.50), the estimate (2.4) holds on $D(s/2) \times \Pi_\alpha$. The similar discussion in [22] shows us that the estimate (2.4) can be extended to the domain $D(s/2, r/2)$. The estimates (2.5) and (2.6) are simple and we omit the details.

Note that Φ is analytic on $D(s/2, r/2)$, we deduce that $H \circ \Phi = N^* + P^*$ is analytic on $D(s/2, r/2)$. We need to prove that $\partial_y P^* = \partial_z P^* = \partial_{\bar{z}} P^* = 0$, $\partial_{z_i z_j}^2 P^* = \partial_{z_i \bar{z}_j}^2 P^* = \partial_{\bar{z}_i \bar{z}_j}^2 P^* = 0$ on $D(s/2) \times \Pi_\alpha$. In the following we only give the proof for $\partial_{z_i \bar{z}_j}^2 P^* = 0$ and omit the proofs for the others.

Note that $\|\partial_{z_i \bar{z}_j}^2 P_\nu\|_{D(s/2)} \leq \varepsilon_\nu$ and $\|\partial_{z_i \bar{z}_j}^2 (P_\nu - P_{\nu+1})\|_{D(s/2)} \preceq \varepsilon_\nu + \varepsilon_{\nu+1} \preceq \varepsilon_\nu$. It then follows

$$\|\partial_{z_i \bar{z}_j}^2 (P_\nu - P^*)\|_{D(s/2)} \leq \sum_{k=\nu}^{\infty} \|\partial_{z_i \bar{z}_j}^2 (P_k - P_{k+1})\|_{D(s/2)} \preceq \varepsilon_\nu$$

and so,

$$\|\partial_{z_i \bar{z}_j}^2 P^*\|_{D(s/2)} \leq \|\partial_{z_i \bar{z}_j}^2 P_\nu\|_{D(s/2)} + \|\partial_{z_i \bar{z}_j}^2 (P_\nu - P^*)\|_{D(s/2)} \preceq \varepsilon_\nu$$

for all ν which means that $\partial_{z_i \bar{z}_j}^2 P^* = 0$ on $D(s/2) \times \Pi_\alpha$. \square

6. MEASURE ESTIMATES

In this section we prove the measure estimates.

Theorem 6.1. *Let ω_ν, Ω_ν for $\nu \geq 0$ be Lipschitz maps on Π satisfying*

$$|\omega_\nu - \omega|, \|\Omega_\nu - \Omega\|_{2\beta} \leq \alpha, \quad |\omega_\nu - \omega|^\mathcal{L}, \|\Omega_\nu - \Omega\|_{2\beta}^\mathcal{L} \leq \frac{1}{2L},$$

and define the sets $\mathcal{R}_{kl}^\nu(\alpha)$ as in Lemma 5.16 choosing $\tau \geq n + 2$. Then

$$\text{Meas}(\Pi \setminus \Pi_\alpha) \leq \text{Meas}\left(\bigcup \mathcal{R}_{kl}^\nu(\alpha)\right) \rightarrow 0, \quad \text{as } \alpha \rightarrow 0.$$

In estimating the measure of the resonance zones it is not necessary to distinguish between the various perturbations ω_ν and Ω_ν of the frequencies, since only the size of the perturbation matters. Therefore, we write ω', Ω' for all of them, and we have

$$|\omega' - \omega|, \|\Omega' - \Omega\|_{2\beta} \leq \alpha, \quad |\omega' - \omega|^\mathcal{L}, \|\Omega' - \Omega\|_{2\beta}^\mathcal{L} \leq \frac{1}{2L}.$$

Similarly, we write \mathcal{R}'_{kl} rather than \mathcal{R}_{kl}^ν for the various resonance zones. The proof of Theorem 6.1 requires a couple of lemmas.

Lemma 6.2. *For $l \in \Lambda = \{l : 1 \leq |l| \leq 2\}$,*

$$\ln(1 + \langle l \rangle) \geq \frac{1}{8} \|l\|_{2\beta}^{\frac{1}{2\beta}} \|l\|_{-2\beta}^{\frac{1}{2\beta}},$$

where $\|l\|_{\pm 2\beta} = \sup_{j \geq 1} |l_j| (1 + \ln j)^{\pm 2\beta}$.

Proof. We only prove the most complicated case, i.e., $l = (\dots, 1, \dots, -1, \dots)$. In other words, $l_i = 1, l_j = -1$ with $i < j$. Set $b = \langle l \rangle = j - i$.

Case 1: $b \geq 2e$. Clearly,

$$\|l\|_{2\beta} \|l\|_{-2\beta} = \left(\frac{1 + \ln j}{1 + \ln i} \right)^{2\beta} \leq (1 + \ln(i + b))^{2\beta}.$$

If $b \geq i$, then $\|l\|_{2\beta} \|l\|_{-2\beta} \leq 2^{2\beta} (\ln b)^{2\beta}$. It follows $\ln \langle l \rangle \geq \frac{1}{2} \|l\|_{2\beta}^{\frac{1}{2\beta}} \|l\|_{-2\beta}^{\frac{1}{2\beta}}$. If $b \leq i$, it follows $i \leq j \leq 2i$. From a straightforward computation,

$$\|l\|_{2\beta} \|l\|_{-2\beta} \leq \left(\frac{1 + \ln 2i}{1 + \ln i} \right)^{2\beta} \leq 2^{2\beta}.$$

We obtain $\ln\langle l \rangle \geq 1 \geq \frac{1}{2} \|l\|_{2\beta}^{\frac{1}{2\beta}} \|l\|_{-2\beta}^{\frac{1}{2\beta}}$.

Case 2: $1 \leq b \leq 2e$. Similarly, $\frac{1}{4} \|l\|_{2\beta}^{\frac{1}{2\beta}} \|l\|_{-2\beta}^{\frac{1}{2\beta}} \leq 1$. It follows

$$\ln(1 + \langle l \rangle) \geq \ln 2 \geq \frac{1}{2} \geq \frac{1}{8} \|l\|_{2\beta}^{\frac{1}{2\beta}} \|l\|_{-2\beta}^{\frac{1}{2\beta}}.$$

For other cases the proofs are similar. \square

Lemma 6.3. *If $\mathcal{R}'_{kl} \neq \phi$ and $k \neq 0$, $l \in \Lambda$, then $|k| \geq c_3 \langle l \rangle$, where $0 < \alpha \leq \min\{1, \frac{1}{2}|a_1|\}$, c_3 is a constant depending on a_1, M, M_1 , where a_1 are defined in Assumption \mathcal{A} .*

Proof. Case 1: $l = (\dots, 1, \dots, -1, \dots)$. In other words, $l_i = 1$, $l_j = -1$, $i < j$.

$$\langle k, \omega' \rangle + \langle l, \Omega' \rangle = \underbrace{(\overline{\Omega}_i - \overline{\Omega}_j)}_{I_1} + \underbrace{(\Omega_i - \overline{\Omega}_i) - (\Omega_j - \overline{\Omega}_j)}_{I_2} + \underbrace{\langle l, \Omega' - \Omega \rangle}_{I_3} + \underbrace{\langle k, \omega' \rangle}_{I_4},$$

where $|I_1| \geq |a_1||i - j| = |a_1|\langle l \rangle$ and $|I_2| \leq 2M_1$ (note $\delta < 0$). From $\|\Omega' - \Omega\|_{2\beta} \leq \alpha$, it follows $|I_3| = |\langle l, \Omega' - \Omega \rangle| \leq \alpha(1 + \ln i)^{-2\beta} + \alpha(1 + \ln j)^{-2\beta} \leq 2\alpha$. On the other hand, $|I_4| \leq |k|(\alpha + M)$. Thus, $|I_2 + I_3 + I_4| \leq 2M_1 + 2\alpha + |k|(\alpha + M)$. If $\mathcal{R}'_{kl} \neq \phi$, then there exists $\xi \in \Pi$ so that

$$|\langle k, \omega'(\xi) \rangle + \langle l, \Omega'(\xi) \rangle| < \frac{\alpha \langle l \rangle}{A_k}.$$

Thus,

$$\begin{aligned} \frac{\alpha \langle l \rangle}{A_k} &> |\langle k, \omega' \rangle + \langle l, \Omega' \rangle| \\ &\geq |I_1| - |I_2 + I_3 + I_4| \\ &\geq |a_1|\langle l \rangle - (2M_1 + 2\alpha + |k|(\alpha + M)). \end{aligned}$$

If $0 < \alpha \leq \min\{1, \frac{1}{2}|a_1|\}$, then $\langle l \rangle \leq \frac{2|k|(2M_1 + M + 3)}{|a_1|}$ or $|k| \geq \frac{|a_1|}{2(2M_1 + M + 3)} \langle l \rangle := \langle l \rangle c_3$. For other cases the proofs are similar. \square

From [38], we have

Lemma 6.4. *If $|k| \geq 8LM\|l\|_{-2\beta}$, then*

$$\text{Meas}(\mathcal{R}'_{kl}(\alpha)) \leq \frac{\alpha c_4}{A_k},$$

where $c_4 = C_n L^n M^{n-1} \rho^{n-1} c_3^{-1}$ with $\rho = \text{diam}(\Pi)$.

Similar as [38], let

$$L_* = \frac{LM}{c_3 C_\beta}, \quad K_* = 8LM \max_{\|l\|_{2\beta} \leq L_*} \|l\|_{-2\beta} + c(\tau, \iota, \delta) \quad (6.1)$$

with $c(\tau, \iota, \delta)$ defined in (6.3) below, we have

Lemma 6.5. *If $|k| \geq K_*$ or $\|l\|_{2\beta} \geq L_*$, $l \in \Lambda$, then for $k \neq 0$,*

$$\text{Meas}(\mathcal{R}'_{kl}(\alpha)) \leq \frac{\alpha c_4}{A_k}.$$

Proof. The proof is followed by Lemma 6.2 and a straightforward computation. \square

Remark 6.6. *The same holds for $k \neq 0, l = 0$.*

Next we consider the “resonance classes” $\mathcal{R}'_k(\alpha) = \bigcup_{l \in \Lambda}^* \mathcal{R}'_{kl}(\alpha)$, where the star indicates that we exclude the finitely many resonance zones with $0 \leq |k| < K_*$ and $0 < \|l\|_{2\beta} < L_*$. Without loss of generality we suppose $-1 \leq \delta < 0$. If $\delta < -1$, then we set $\delta = -1$.

Lemma 6.7. $Meas(\mathcal{R}'_k(\alpha)) < \frac{c_5 \alpha^\mu}{|k|^{\tau-1}}$, where $\mu = \frac{\delta}{\delta-1}$.

Proof. Write $\Lambda = \Lambda^+ \cup \Lambda^-$, where Λ^- contains those $l \in \Lambda$ with two nonzero components of the opposite sign, and Λ^+ contains the rest. It is easy to obtain $Meas(\bigcup_{l \in \Lambda^+}^* \mathcal{R}'_{kl}(\alpha)) \leq \frac{c_6 |k|^2 \alpha}{A_k}$.

Now we turn to the minus case. For $l \in \Lambda^-$, we have $\langle l, \Omega' \rangle = \Omega'_i - \Omega'_j$ and $\langle l \rangle = |i - j|$, and up to an irrelevant sign, l is uniquely determined by two integers $i \neq j$. We may suppose that $i - j = b > 0$. Then, for $|k| \geq K_* \geq ((\tau + 1)\iota + 1)!$,

$$\begin{aligned} |\langle k, \omega' \rangle + a_1 b| &< \frac{\alpha b}{A_k} + 2\alpha(1 + \ln j)^{-2\beta} + 2M_1 j^\delta \\ &\leq \frac{\alpha b}{|k|^\tau} + 2\alpha(1 + \ln j)^{-2\beta} + 2M_1 j^\delta. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{R}'_{kij}(\alpha) &= \left\{ \xi : |\langle k, \omega'(\xi) \rangle + \langle l, \Omega'(\xi) \rangle| < \frac{\alpha b}{A_k} \right\} \\ &\subset Q_{kbj} := \left\{ \xi : |\langle k, \omega'(\xi) \rangle + a_1 b| < \frac{\alpha b}{|k|^\tau} + 2\alpha(1 + \ln j)^{-2\beta} + 2M_1 j^\delta \right\}. \end{aligned}$$

Moreover, $Q_{kbj} \subset Q_{kbj_0}$ for $j \geq j_0$. For fixed $b \leq c_3^{-1}|k|$, we obtain

$$\begin{aligned} Meas\left(\bigcup_{i-j=b}^* \mathcal{R}'_{kij}(\alpha)\right) &\leq \sum_{j < j_0} Meas(\mathcal{R}'_{kij}(\alpha)) + Meas(Q_{kbj_0}) \\ &\leq \frac{c_4 \alpha j_0}{A_k} + c_6 \left(\frac{\alpha b}{|k|^{\tau+1}} + \frac{2\alpha(1 + \ln j_0)^{-2\beta}}{|k|} + \frac{2M_1 j_0^\delta}{|k|} \right). \end{aligned} \quad (6.2)$$

Choose $j_0 = \max\{\exp(|k|^{\frac{\tau-1}{2\beta}}), \alpha^{\frac{\gamma}{\delta}} |k|^{\frac{1-\tau}{\delta}}\}$, where γ will be fixed in the end. By computation, if choose

$$|k| \geq K_* \geq c(\tau, \iota, \delta) := \max\left\{2^{2\iota}, \left(\iota(\tau + 1 + \frac{1-\tau}{\delta}) + 1\right)!\right\}, \quad (6.3)$$

then

$$(6.2) \leq c_7 \left(\frac{\alpha^{1+\frac{\gamma}{\delta}}}{|k|^\tau} + \frac{\alpha}{|k|^\tau} + \frac{\alpha^\gamma}{|k|^\tau} \right).$$

Note $0 < -\delta \leq 1$, if choose $\gamma = \frac{\delta}{\delta-1}$, then (6.2) $\leq c_8 \frac{\alpha^\mu}{|k|^\tau}$. Summing over b , $Meas(\bigcup_{l \in \Lambda^-}^* \mathcal{R}'_{kl}(\alpha)) \leq c_9 \frac{\alpha^\mu}{|k|^{\tau-1}}$. Thus

$$Meas\left(\bigcup_{l \in \Lambda}^* \mathcal{R}'_{kl}(\alpha)\right) \leq \frac{c_6 |k|^2 \alpha}{A_k} + c_9 \frac{\alpha^\mu}{|k|^{\tau-1}} \leq c_5 \frac{\alpha^\mu}{|k|^{\tau-1}}.$$

□

From Remark 6.6, if $k \neq 0$, $l_0 = 0$, $Meas(\mathcal{R}'_{kl_0}(\alpha)) \leq \frac{c_4 \alpha}{A_k}$, where we define $\mathcal{R}''_k(\alpha) = \bigcup_{|k| \geq K_*} \mathcal{R}'_{kl_0}(\alpha)$. Note the choice of K_* and $|k| \geq K_*$, we deduce that for $l_0 = 0$, $Meas(\mathcal{R}'_{kl_0}(\alpha)) \leq \frac{c_4 \alpha}{|k|^{\tau-1}} \leq \frac{c_4 \alpha^\mu}{|k|^{\tau-1}}$. Thus we have

Lemma 6.8. For $|k| \geq K_*$, $Meas(\mathcal{R}'_k(\alpha) \cup \mathcal{R}''_k(\alpha)) \leq \frac{c_{10} \alpha^\mu}{|k|^{\tau-1}}$.

Lemma 6.9. *There exists a finite subset $\mathcal{X}_1 \subset \mathcal{Z}$ and a constant \tilde{c}_1 such that*

$$\text{Meas}\left(\bigcup_{(k,l) \notin \mathcal{X}_1} \mathcal{R}_{kl}^\nu(\alpha)\right) \leq \tilde{c}_1 \rho^{n-1} \alpha^\mu$$

for all sufficiently small α . The constant \tilde{c}_1 and the index set \mathcal{X}_1 are monotone functions of the domain Π : they do not increase for closed subsets of Π . In particular,

$$\mathcal{X}_1 \subset \{(k, l) : 0 \leq |k| < \tilde{K}_* := 16LM + c(\tau, \iota, \delta), 0 < \|l\|_{2\beta} \leq L_*, l \in \Lambda\}.$$

Proof. The proof is from Lemma 6.7 with $\tau \geq n + 2$ and similar as [38]. \square

If as above we set $l_0 = 0$, then we obtain a similar lemma as above.

Lemma 6.10. *There exists a finite subset $\mathcal{X}_2 \subset \mathcal{Z}$ and a constant \tilde{c}_2 such that*

$$\text{Meas}\left(\bigcup_{(k,l_0) \notin \mathcal{X}_2} \mathcal{R}_{kl_0}^\nu(\alpha)\right) \leq \tilde{c}_2 \rho^{n-1} \alpha^\mu$$

for all sufficiently small α . The constant \tilde{c}_2 and the index set \mathcal{X}_2 are monotone functions of the domain Π : they do not increase for closed subsets of Π . In particular, $\mathcal{X}_2 \subset \{(k, l_0) : 0 \leq |k| < \tilde{K}_\}$.*

Proof of Theorem 6.1: If we choose

$$\gamma_0 \leq \min \left\{ \left(\frac{1}{4c_1} \right)^{10}, \left(\frac{1}{8c_0M} \right)^4, c_2(s_0), \frac{1}{4c_1} \exp(-2\tilde{K}_*^{\frac{1}{2}}) \right\},$$

then $K_0 \geq \tilde{K}_*$. Thus, when $\nu \geq 1$ and $l \in \Lambda$, $\text{Meas}\left(\bigcup_{(k,l) \in \mathcal{X}_1} \mathcal{R}_{kl}^\nu(\alpha)\right) = 0$. Since \mathcal{X}_1 is finite, Assumption \mathcal{A} implies $\text{Meas}\left(\bigcup_{(k,l) \in \mathcal{X}_1} \mathcal{R}_{kl}^\nu(\alpha)\right) \rightarrow 0$ as $\alpha \rightarrow 0$. Combined with Lemma 6.9, we have

$$\text{Meas}\left(\bigcup_{l \in \Lambda} \mathcal{R}_{kl}^\nu(\alpha)\right) \rightarrow 0, \quad \text{as } \alpha \rightarrow 0. \quad (6.4)$$

The proof for $l_0 = 0$ is similar. We have

$$\text{Meas}\left(\bigcup_{k \neq 0} \mathcal{R}_{kl_0}^\nu(\alpha)\right) \rightarrow 0, \quad \text{as } \alpha \rightarrow 0. \quad (6.5)$$

Combined with (6.4) and (6.5) we complete the proof. \square

7. APPENDIX

Lemma 7.1. *For $j \geq 1$ and $\beta > 1$, there exists a constant $C(\beta)$ independent of j such that,*

$$\sum_{l \geq 1} \frac{1}{(1 + |j - l|)(1 + \ln l)^\beta} \leq C(\beta).$$

Proof. The summation is divided into three parts,

$$\left(\sum_{1 \leq l \leq j/2} + \sum_{j/2 < l \leq j} + \sum_{l > j} \right) \frac{1}{(1 + |j - l|)(1 + \ln l)^\beta} := (I) + (II) + (III).$$

Hence the result is followed by the following facts:

$$\begin{aligned}
(I) &\leq \frac{j}{2} \frac{1}{\frac{j}{2}(1+\ln 2)^\beta} \leq C(\beta), \\
(II) &\leq \sum_{1 \leq k \leq j/2} \frac{1}{k(1+\ln(j-k-1))^\beta} \leq \sum_{1 \leq k \leq j/2} \frac{1}{k(1+\ln k)^\beta} \leq C(\beta), \\
(III) &\leq \sum_{k>1} \frac{1}{(1+k)(1+\ln(j+k))^\beta} \leq \sum_{k>1} \frac{1}{(1+k)(1+\ln k)^\beta} \leq C(\beta).
\end{aligned}$$

□

Remark 7.2. If $\beta \geq 2$, then $\sum_{l \geq 1} \frac{1}{(1+|j-l|)(1+\ln l)^\beta} \leq C$, where C is independent of j and β .

Lemma 7.3. For any $j, l \geq 1$, $p \geq 2$ and $\beta \geq 1$,

$$\sum_{l \geq 1} \frac{(1+j)^2}{l^p(1+|j-l|)^2(1+\ln l)^{2\beta}} \leq C.$$

Proof.

$$\begin{aligned}
&\sum_{l \geq 1} \frac{(1+j)^2}{l^p(1+|j-l|)^2(1+\ln l)^{2\beta}} \\
&\leq \sum_{2l \leq j} \frac{(1+j)^2}{l^p(1+|j-l|)^2(1+\ln l)^{2\beta}} + \sum_{2l > j} \frac{(1+j)^2}{l^p(1+|j-l|)^2(1+\ln l)^{2\beta}} = (I) + (II).
\end{aligned}$$

We estimate (I) and (II) respectively. For (I), note $|j-l| \geq j/2$ and $p \geq 2$, we have

$$(I) \leq \sum_{2l \leq j} \frac{(1+j)^2}{l^p(1+j/2)^2} \leq 4 \sum_{l \geq 1} \frac{1}{l^p} \leq C.$$

For (II), from $p \geq 2$ and $\beta \geq 1$,

$$\begin{aligned}
(II) &\leq \sum_{2l > j} \frac{(1+j)^2}{l^2(1+|j-l|)^2(1+\ln l)^{2\beta}} \\
&\leq \sum_{2l > j} \frac{(1+j)^2}{(j/2)^2(1+|j-l|)^2(1+\ln l)^{2\beta}} \\
&\leq C \sum_{l \geq 1} \frac{1}{(1+|j-l|)^2(1+\ln l)^{2\beta}} \\
\text{Lemma 7.1} &\leq C.
\end{aligned}$$

□

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